DYNAMIC SYSTEMS WITH THREE VARIABLE PARAMETERS

by

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UNITED STATES
NAVAL POSTGRADUATE SCHOOL

THESIS

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ABSTRACT

A study of Dynamic Systems with three variable parameters is made by an initial scanning of the basic geometric properties on the three dimensional space generated by these parameters. From these geometric properties, a Root-locus technique that simplifies greatly the amount of work in analysis and design is developed. This technique is extended to systems with k variable parameters. Finally, singular lines in the parameter plane are treated as a special case of the parameter space, and formulae are derived for a fourth order system leaving the field open for systems of higher order.

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1. INTRODUCTION OF THE THIRD PARAMETER

This study takes as a starting point Mitrovic's approach to the root-finding problem. Although this is essentially a two-variable method which will yield only two relations out of the three needed for the three-parameter case, the parametric equations obtained will help in understanding the nature of the basic geometric properties of the parameter space generated by α, β, and γ, the three variable parameters. The fact that it is possible to obtain only two relations is justified because a point on the s-plane is completely defined by only two conditions, and therefore any other possible relations will be linearly related to the two originated by Mitrovic's[2] root-finding method.

Let us now describe briefly Mitrovic's approach, as a base for the present study:

The characteristic equation for a given system can be written in a general form as:

$$F(S) = a_n S^n + a_{n-1} S^{n-1} + \cdots\cdots + a_1 S + a_o = 0 \qquad 1.1$$

or in a closed form as:

$$F(S) = \sum_{k=0}^{n} a_k S^k = 0 \qquad 1.2$$

If now S is expressed as:

$$S = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \qquad 1.3$$

where ζ and ω_n are defined in figure 1.1, which is the representation of a point in the s-plane that is defined by assigning values to ζ and ω_n.

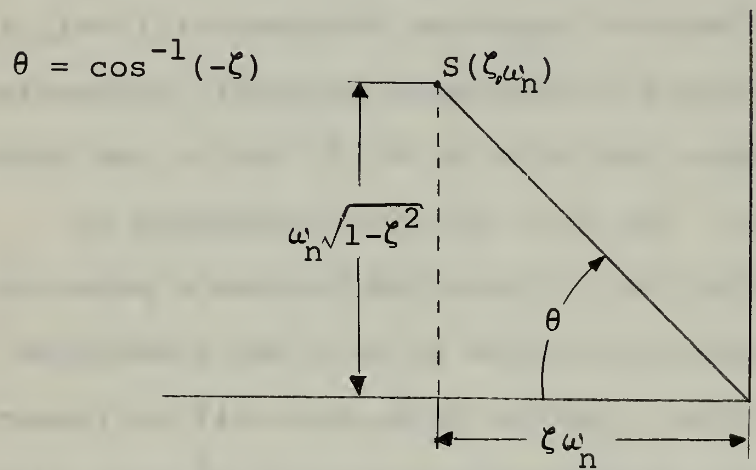


Fig. 1.1 - Representation of a point on the s-plane

Equation 1.3, written in another form yields:

$$S = \omega(\cos\theta + j\sin\theta) = \omega e^{j\theta} \quad 1.4$$

where $\theta = \cos^{-1}(-\zeta)$. Then S^k can be written:

$$S^k = \omega^k e^{jk\theta} = \omega^k(\cos k\theta + j\sin k\theta), \quad 1.5$$

from which the Chebyshev functions are obvious and are:

$$T_k(\zeta) = \cos k\theta = \cos(k\cos^{-1}\zeta) \quad 1.6$$

$$\overline{U}_k(\zeta) = \frac{\sin k\theta}{\sin\theta} = \frac{\sin(k\cos^{-1}\zeta)}{\sin(\cos^{-1}\zeta)} \quad 1.7$$

$$s^k = \omega^k\left[T_k(-\zeta) + j\sqrt{1-\zeta^2}\, U_k(-\zeta)\right] \qquad 1.8$$

$$= \omega^k\left[(-1)^k\, T_k(\zeta) + j\sqrt{1-\zeta^2}(-1)^{k+1} U_k(\zeta)\right] \qquad 1.9$$

Inserting these in 1.2 and requiring that the reals and imaginaries go to zero independently, provides the two equations:

$$\sum_{k=0}^{n} a_k \omega_n{}^k (-1)^k T_k(\zeta) = 0$$

$$\sum_{k=0}^{n} a_k \omega_n{}^k (-1)^{k+1} U_k(\zeta) = 0 \qquad 1.10$$

But $T_k(\zeta) = \zeta(U_k(\zeta)) - (U_{k-1}(\zeta))$

and upon substitution, the following equations are obtained:

$$\sum_{k=0}^{n} (-1)^k a_k \omega_n{}^k\, U_{k-1}(\zeta) = 0$$

$$\sum_{k=0}^{n} (-1)^k a_k \omega_n{}^k\, U_k(\zeta) = 0 \qquad 1.11$$

where U_k can be obtained from the recurrence formula:

$$U_{k+1}(\zeta) = 2\zeta\, U_k(\zeta) - U_{k-1}(\zeta) \qquad 1.12$$

where $U_o(\zeta) = 0$, and $U_1(\zeta) = 1$

Now, the coefficients are defined to be:

$$a_k = \alpha b_k + \beta c_k + \gamma d_k + f_k \qquad 1.13$$

where α, β, and γ are the independent variable parameters of the system under analysis. By substituting this expression for a_k in equations 1.11, these equations can be rewritten as follows:

$$\alpha B_1 + \beta C_1 + \gamma D_1 + F_1 = 0$$
$$\alpha B_2 + \beta C_2 + \gamma D_2 + F_2 = 0 \qquad 1.14$$

where

$$B_1 = \sum_{k=0}^{n} (-1)^k b_k \ \omega_n^k U_{k-1}(\zeta) \qquad B_2 = \sum_{k=0}^{n} (-1)^k b_k \ \omega_n^k U_k(\zeta)$$

$$C_1 = \sum_{k=0}^{n} (-1)^k c_k \ \omega_n^k U_{k-1}(\zeta) \qquad C_2 = \sum_{k=0}^{n} (-1)^k c_k \ \omega_n^k U_k(\zeta)$$

$$D_1 = \sum_{k=0}^{n} (-1)^k d_k \ \omega_n^k U_{k-1}(\zeta) \qquad D_2 = \sum_{k=0}^{n} (-1)^k d_k \ \omega_n^k U_k(\zeta) \qquad 1.15$$

$$F_1 = \sum_{k=0}^{n} (-1)^k f_k \ \omega_n^k U_{k-1}(\zeta) \qquad F_2 = \sum_{k=0}^{n} (-1)^k f_k \ \omega_n^k U_k(\zeta)$$

Information concerning the design properties of the system must be obtained from these expressions (equations 1.14). This information can be:

a). For a given set of Performance Specifications, what values of α, β, and γ, the variable parameters of the system, are needed in order to obtain the dominance conditions that will allow us to meet the given performance specifications?

b). The physical meaning of the values of α, β, and γ, whether or not these values are actually attainable, and if not, what modifications must be introduced to the system, such that it is possible to obtain the desired values.

c). Existence of dominance conditions, if these exist, and the locations of the remaining roots on the s-plane, and their influence in the behavior of the system.

If ζ and ω_n are given, then the set of equations 1.14, will contain three unknowns: α, β, and γ. Therefore the surfaces generated on the α, β, γ coordinate system can be expressed only in a parametric form, generating sets of parameter-planes in α and β for a given value of γ; or it is possible to obtain a three dimensional representation which is not a practical tool for engineering purposes.

2. REAL ROOTS

The characteristic equation for a system can be rewritten as

$$\alpha A(S) + \beta B(S) + \gamma C(S) + D(S) = 0 \qquad 2.1$$

By making $S = -\sigma_1$, A(S), B(S), C(S) and D(S), become real numbers A, B, C and D, so that the above equation becomes

$$\alpha A + \beta B + \gamma C + D = 0 \qquad 2.2$$

which is the equation of a plane on the system of coordinates generated by α , β, and γ, with intercepts as follows:

$$\begin{array}{lll} \text{on the } \alpha \text{ axis} & \alpha = -D/A & \\ \text{on the } \beta \text{ axis} & \beta = -D/B & 2.3 \\ \text{on the } \gamma \text{ axis} & \gamma = -D/C & \end{array}$$

This is, then the first set of known surfaces that can be used in the three dimensional space generated by α, β, and γ. However, there is need for a graphical representation which allows the determination of the desired information without tedious use of computarized data, with which it is hard to get a clear vision of the trend of the different values for changes in the parameters of the system. In order to obtain this graphical representation, it is possible to use descriptive geometry as a useful tool for the three-dimensional analysis.

The construction rules are simple and easy to retain and with some practice it can be a tool that can be used very often in this study.

The first step is the representation of a plane in a three-dimensional space: A plane is completely defined if two lines belonging to such a plane are defined. It is possible to draw these two lines by determining the intercepts of the plane which represents $S = -\sigma_1$ in the α, β, γ space, with the $\alpha\beta$ and $\alpha\gamma$ planes. (See figure 2.1a which is a representation of the plane $S = -\sigma_1$ on the first octant of the coordinates axes). Now, by rotating the γ axis in the plane $\beta\gamma$ in the manner shown in figure 2.1a, and in amount such that the three axes lie in the plane generated by the axes α and β, figure 2.1b is obtained, which contains all the information concerning the three-dimensional graph. However, this time it is possible to work with real values and to plot all the information in one plane. Of course, once it is learned how to work with this construction method it will not be necessary to have the three-dimensional graphs. Let us assume we are to investigate whatever values of α and β, are obtainable with given values of $S = -\sigma_1$ and $\gamma = \gamma_1$. In the three-dimensional picture, we would pass a plane parallel to the plane $\alpha\beta$, through the given value of $\gamma = \gamma_1$ in order to obtain the plane PNM. As can be seen, this is not a very practical method even if there exists the possibility of building an actual three-dimensional representation of the plane $S = -\sigma_1$. By the construction method proposed, what has to be done is to project the chosen value of γ

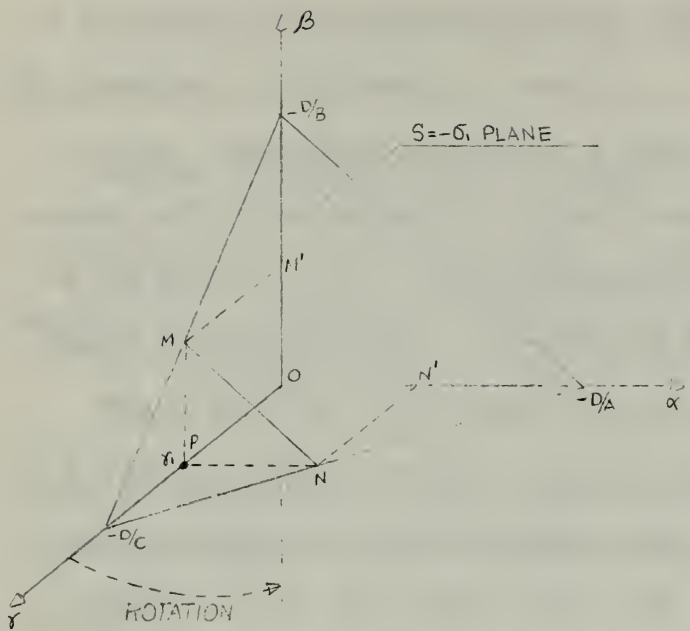


Fig. 2.1a

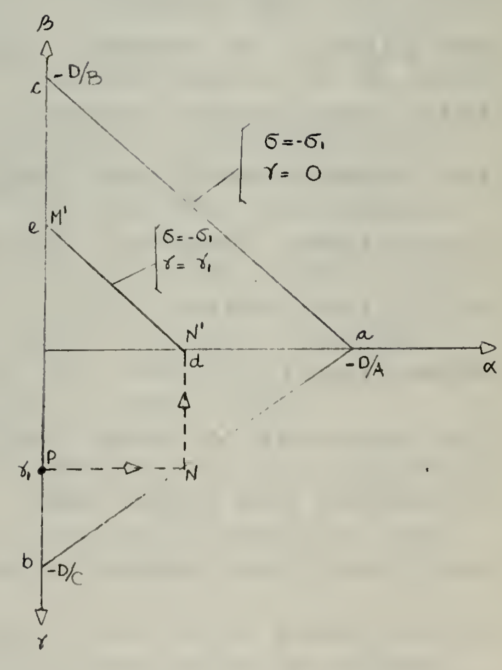


Fig. 2.1b

Fig. 2.1a - Representation of a plane in three dimensions

Fig. 2.1b - Representation of a plane using projections and all coordinate axes in one plane onto the line ab, determine the point N, and project this point N onto N', and finally draw a parallel line to ac through N'. Then the line de will be the actual representation of the line MN of the three-dimensional graph, from which we can read the values of α and β. This construction method can be more easily understood by the use of a simple example:

<u>Example 2.1</u>: Let us assume that a system has the following characteristic equation

$$S^4 + (3\alpha+4\beta)S^3 + (6\alpha+3\beta)S^2 + (10\beta+50\gamma)S + (5\alpha+5\gamma) = 0 \qquad 2.4$$

which can be rearranged in the form of equation 2.2 as follows:

$$G(S) = (3S^3+6S^2+5)\alpha + (4S^3+3S^2+10S)\beta + (50S+5)\gamma+S^4 = 0$$

If $S = -5$ is to be investigated, then for this value of S:

$$G(S) = -220\alpha - 475\beta - 245\gamma + 625 = 0$$

and:

the α intercept $= -\dfrac{625}{-220} = 2.84$

the β intercept $= -\dfrac{625}{-475} = 1.315$

and the γ " $= -\dfrac{625}{-245} = 2.55$

Figure 2.2 is the representation of the positive axes α, β, and γ. The negative axes can also be identified by labeling the β axis, as the $-\gamma$ axis, the γ axis as the $-\beta$ axis. Lines ab and ac are the intercepts of the plane $S = -5$ with the $\alpha\beta$ and $\alpha\gamma$ planes; dotted lines are extensions of

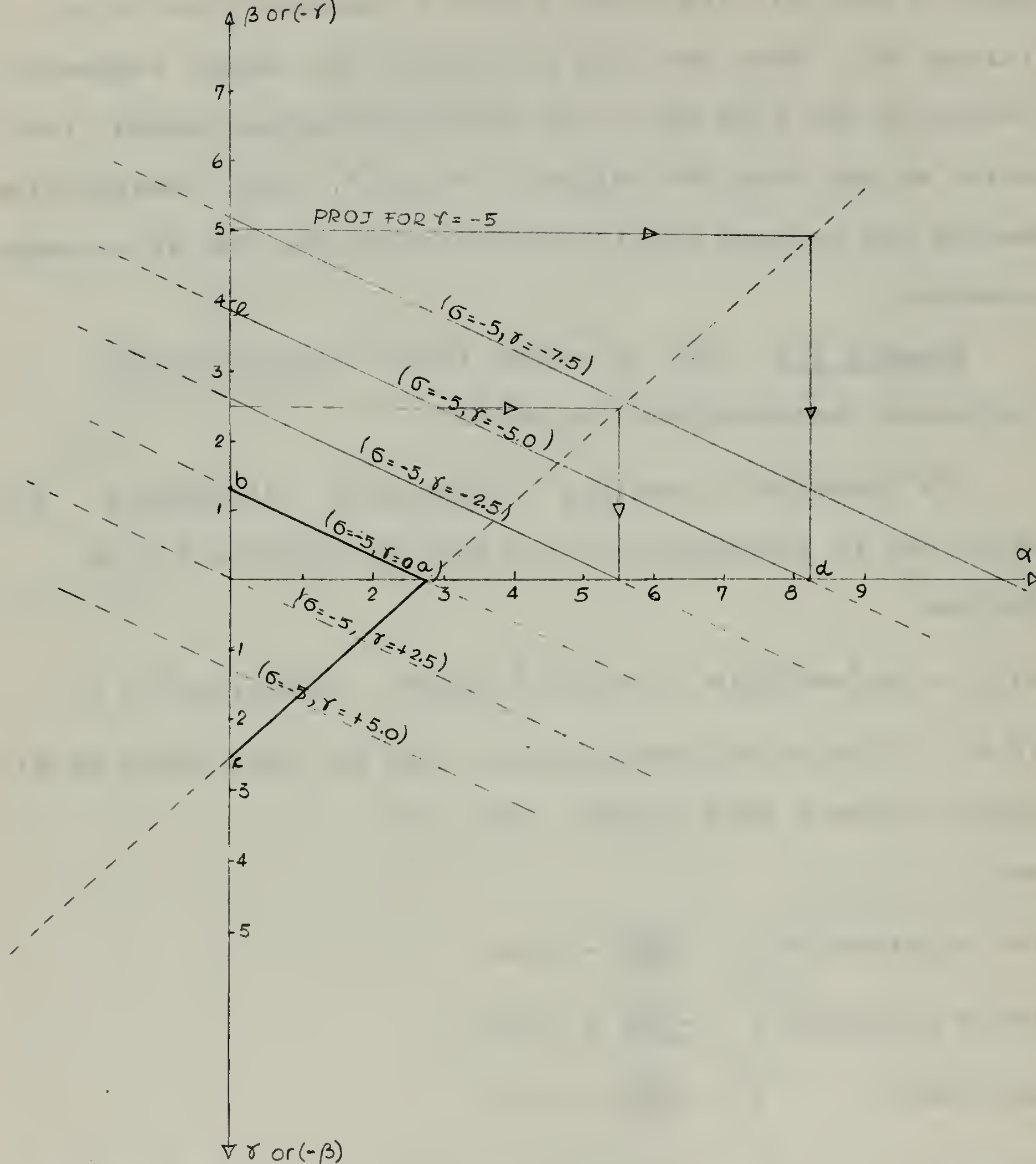


Fig. 2.2 - S = -5 lines in the $\alpha\beta\gamma$ space for Example 2.1

the lines on the negative side of the axes. Let us assume that we wish to investigate what values of α and β are suitable for a given value of γ, say $\gamma = -5$. By setting $\gamma = -5$ and projecting this value of γ onto line ac, point d is obtained from which a line parallel to ab is drawn. This is line de, and this line contains the information about the values of α and β that will satisfy equation 2.4 for S = -5 and $\gamma = -5$. As a check, let us choose the point f; at this point $\alpha = +5$, $\beta = +1.58$ and $\gamma = -5$, which satisfies equation 2.4 if S = -5.

Another interesting feature of this construction method is that due to the linearity properties, once the "advance" over the α axis for a given change in γ is determined, a grid can be easily drawn without the need of projecting each value of γ, by just advancing equal distances on the α axis and drawing lines parallel to ab. The loci for $\gamma = -5.0$, $\gamma = -7.5$, $\gamma = +2.5$ and $\gamma = 5.0$ were drawn by this method. Similar results can be obtained by determining the "advance" in the β axis. In summary, all the necessary information is: the intercepts of the plane with the axes, the "advance" for a desired increment of one of the parameters. Then the surface generated by S = -5 will be completely described. Therefore, for a given value of $S = -\sigma_1$, the graph will look like the one in figure 2.3. When several values of σ are needed, confusion can be avoided by identifying each pair of lines by the common point on the α axis (See figure 2.3).

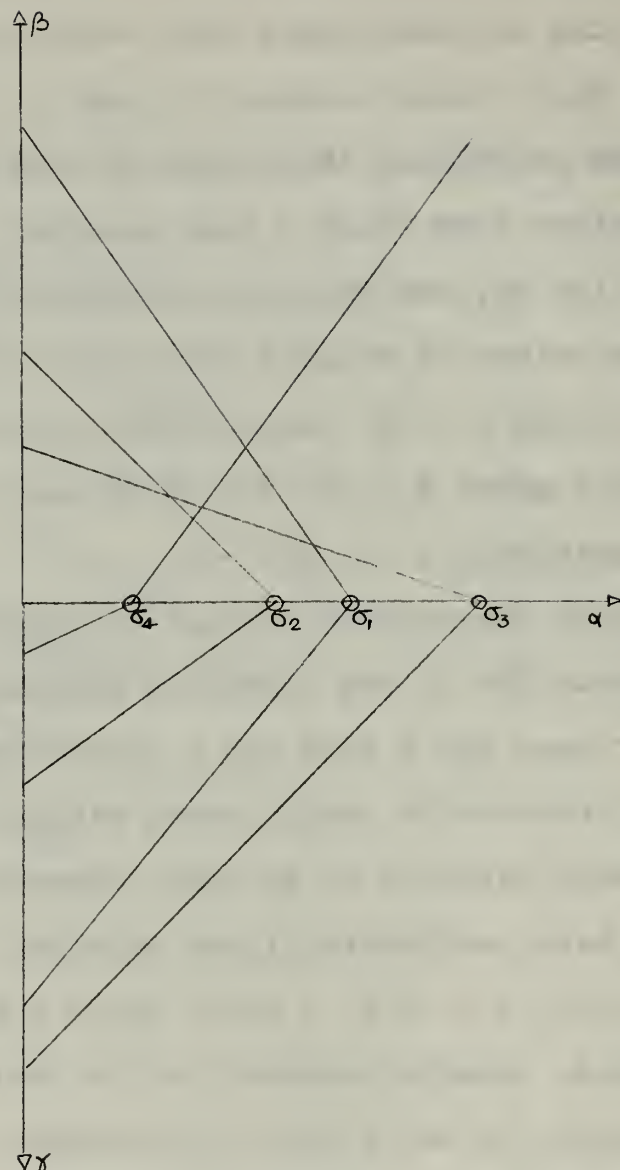


Fig. 2.3 - Grid of loci for several values of real roots in the $\alpha\beta\gamma$ space.

This method of construction will be useful in cases in which it is desired to analyze well damped systems and also as an easy way to construct a grid of values of real roots, which can be superimposed over a previously determined set of constant ζ curves, in order to determine the possible values of real roots expected for given values of α, β, γ, ζ and ω_n.

Example 2.2: Consider a voltage regulator. The compensation schemes have been worked out and this example will be analyzed through this study so we can obtain a better picture of the use of this construction method.

After manipulation of the block diagram the characteristic equation of the system is:

$$S^4 + (2x10^6K_{10}+1350)S^3 + (3x10^8K_{10}+4x10^8K_{11}+3.85x10^5)S^2$$
$$+ (10^{10}K_{10}+2x10^{10}K_{11}+36x10^6)S + 2x10^{10}K_aK_p+2.1x10^{10} = 0$$

by letting $K_{10} = \alpha$, $K_{11} = \beta$, $K_aK_p = \gamma$, and rearranging the characteristic equation in the form of equation 2 2, it becomes:

$$(2x10^6S^3+3x10^8S^2+10^{10}S)\alpha + (4x10^8S^2+2x10^{10}S)\beta +$$
$$2x10^{10}\gamma+S^4+1350S^3+3.85x10^5S^2+36x10^6S+2.1x10^{10} = 0$$

For S = -1 the equation of the plane is:

$$-9702\alpha - 19600\beta + 20,000\gamma + 20964.38 = 0$$

where the α intercept $= - \dfrac{20,964.38}{-9702} = 2.16,$

the β intercept $= - \dfrac{20,964.38}{-19,600} = 1.06,$

and the γ intercept $= - \dfrac{20,964.38}{20,000} = -1.04.$

For S = -5, the equation of the plane is:

$$-4275\alpha - 9000\beta + 2000\gamma + 2082.95 = 0$$

the α intercept $= - \dfrac{2082.95}{-4275} = 0.488,$

the β intercept $= - \dfrac{2082.95}{-9000} = 0.231,$

and the γ intercept $= - \dfrac{2082.95}{2000} = -1.04.$

The results of this problem can be easily checked on figure 2.4 for the case let us take say $\alpha = 0.5$, $\beta = 0.5$ by drawing a line, parallel to ab through the point ($\alpha = 0.5$, $\beta = 0.5$), we get point c on the α axis, and then by projecting this point onto the line corresponding to $\sigma = -1.0$ on the plane $\alpha\gamma$, we get $\gamma = -0.315$ these values satisfy the characteristic equation of the system for S = -1.0. Another variation can be obtained by selecting a value of γ, this is done for $\sigma = -5.0$ in which $\gamma = 1.5$ was selected, by projecting and drawing a parallel line to de we obtain the line labeled ($\sigma = -5.0$, $\gamma = 1.5$). Any pair of values for α and β on this line will satisfy the characteristic equation for the system. We have picked the point B, which yields $\alpha = 2.0$ and $\beta = -0.385$ which checks.

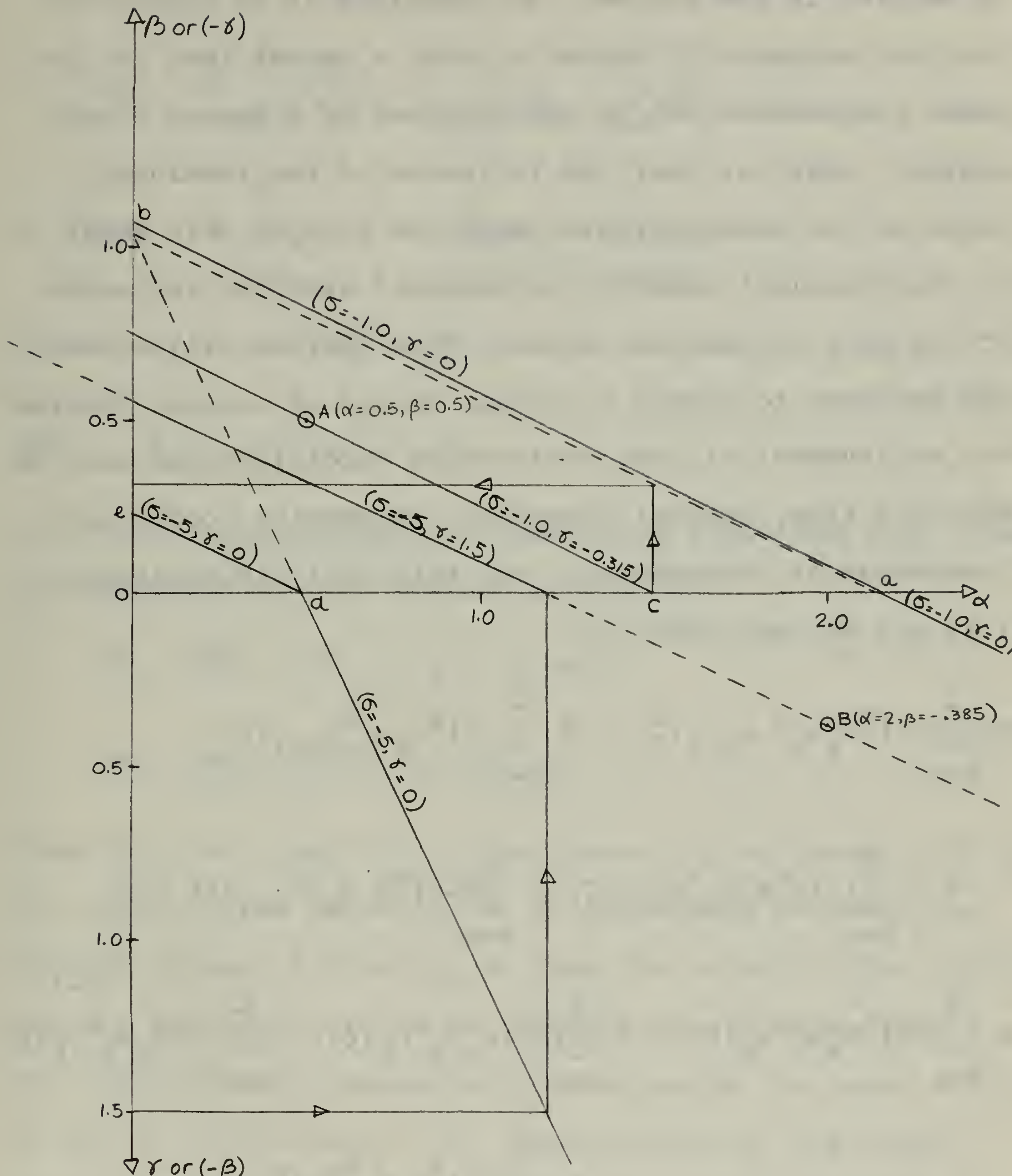


Fig. 2.4 - S = -5.0 and S = -1.0 loci for Example 2.1

3. COMPLEX CONJUGATE ROOTS

Usually dominance of a certain pair of complex roots is desired in the system. By dominance it is understood that the system will behave in such a manner that its dynamic performance can be approximated by a second order system; that is, that the influence of the remaining roots of the characteristic equation will be very small on the transient behavior as compared with the influence of the pair of dominant poles. This feature will allow the designer to obtain a certain degree of control towards the achievement of some performance specifications such as settling time, natural frequency and damping coefficient (overshoot or undershoot). For this analysis equations 1.11 can be rewritten as:

$$\alpha \sum_{k=0}^{n}(-1)^k b_k \omega_n^k U_{k-1}(\zeta) + \beta \sum_{k=0}^{n}(-1)^k c_k \omega_n^k U_{k-1}(\zeta) + \gamma \sum_{k=0}^{n}(-1)^k d_k \omega_n^k U_{k-1}(\zeta) + \sum_{k=0}^{k}(-1)^k e_k \omega_n^k U_{k-1}(\zeta) = 0$$

$$\alpha \sum_{k=0}^{n}(-1)^k b_k \omega_n^k U_k(\zeta) + \beta \sum_{k=0}^{n}(-1) c_k \omega_n^k U_k(\zeta) + \gamma \sum_{k=0}^{n}(-1)^k d_k \omega_n^k U_k(\zeta) + \sum_{k=0}^{n}(-1)^k e_k \omega_n^k U_k(\zeta) = 0 \tag{3.1}$$

where a_k has been replaced by

$$a_k = \alpha b_k + \beta c_k + \gamma d_k + e_k. \tag{3.2}$$

By assuming given values of ζ and ω_n for a desired dominance condition, equations 3.1 can be written as:

$$\alpha \sum_{k=0}^{n} b_k D_k + \beta \sum_{k=0}^{n} c_k D_k + \gamma \sum_{k=0}^{n} d_k D_k + \sum_{k=0}^{n} e_k D_k = 0$$

$$\alpha \sum_{k=0}^{n} b_k E_k + \beta \sum_{k=0}^{n} c_k E_k + \gamma \sum_{k=0}^{n} d_k E_k + \sum_{k=0}^{n} e_k E_k = 0 \qquad 3.3$$

where $D_k = (-1)^k \omega_n^{\,k}\, U_{k-1}(\zeta)$

$$3.4$$

and $E_k = (-1)^k \omega_n^{\,k}\, U_k(\zeta)$

By evaluating the quantities under the summation sign these two equations can be further reduced to:

$$\alpha B_1 + \beta C_1 + \gamma D_1 + E_1 = 0, \text{ and}$$

$$3.5$$

$$\alpha B_2 + \beta C_2 + \gamma D_2 + E_2 = 0.$$

These are the equations of two planes in the space α, β, γ. The locus of the values of α, β and γ which satisfy the desired values of ζ and ω_n is then the straight line defining the intercept of the two planes. It is known that a real root can be represented as a plane in the space α β γ, so now we can determine the intercept of the real-root plane with the straight line defined by the intercept of the planes described by equations 3.5. There are two possibilities: a) the line is not contained on the plane and then the intersection will be a point in the space α β γ; or b) the line is contained on the plane and then we have a set of values of $\alpha\ \beta\ \gamma$ on the line that will satisfy the desired conditions. In other words, it will be possible to solve a set of three equations with three unknowns as follows:

From equation 3.5

$$\begin{aligned} \alpha B_1 + \beta C_1 + \gamma D_1 + E_1 &= 0 \\ \alpha B_2 + \beta C_2 + \gamma D_2 + E_2 &= 0 \end{aligned} \qquad 3.6$$

and from equation 2.2

$$\alpha A + \beta B + \gamma C + D = 0$$

By Cramer's rule:

$$\alpha = \frac{-E_1(C_2C-BD_2) + E_2(C_1C-BD_1) - D(C_1D_2-C_2D_1)}{\Delta},$$

$$\beta = \frac{B_1(-E_2C+DD_2) - B_2(-E_1C+D\,D_1) + A(-E_1D_2+E_2D_1)}{\Delta}, \qquad 3.7$$

and

$$\gamma = \frac{B_1(-C_2D+B\,E_2) - B_2(-C_1D+BE_1) + A(-C_1E_2+C_2E_1)}{\Delta}.$$

where $\Delta = B_1(C_2C-BD_2) - B_2(C_1C-BD_1) + A(C_1D_2-C_2D_1)$

Let us now illustrate this use of known surfaces in the $\alpha\ \beta\ \gamma$ space by an example:

Assume a third-order system with the following characteristic equation:

$$S^3 + (3\alpha+4\beta)S^2 + (4\alpha+3\beta)S + 2\alpha + \gamma = 0 \qquad 3.8$$

Assume now that we wish a pair of complex poles with second-order dominance at $S = -5 \pm 8.66j$ and a third pole at $S = -50$. What values of α, β and γ must we use to obtain the desired pole values?

By applying $S = -50.0$ to equation 3.8 we obtain:

$$7{,}302\alpha + 10{,}000\beta - 149\gamma - 125{,}000 = 0 \qquad 3.9$$

and by substituting $S = -5 \pm 8.66j$ in equation 3.8 we obtain:

$$\begin{aligned} &298\alpha + 400\beta - \gamma - 1{,}000 = 0 \\ \text{and} \quad &260\alpha + 400\beta - 30\gamma = 0 \end{aligned} \qquad 3.10$$

By solving the system of equations formed by 3.9 and 3.10 we get values of

$$\alpha = 7200, \quad \beta = -5{,}385 \quad \text{and} \quad \gamma = -9{,}400$$

By substituting these values in equation 3.8, we get

$$\begin{aligned} &S^3 + 60S^2 + 600 + 500 = 0 \\ \text{or} \quad &(S + 50)(S^2 + 10S + 100) = 0 \qquad 3.11 \\ \text{or} \quad &(S + 50)(S + 5 - 8.66j)(S + 5 + 8.66j) = 0 \end{aligned}$$

We see that we have three degrees of freedom and therefore have the opportunity of choosing three of the roots of the characteristic equation for systems of order higher than three, once we have made our choice of three of the roots. The remaining roots will depend on the values of α, β and γ (except in the very special case of situation previously described, in which we can find another root by determining the intersection of the line with the plane generated by this extra root). Therefore, we have the freedom of choosing three of the roots of the polynomial. However, for polynomials of a degree higher than three, this method does not tell us anything about the remaining roots of the polynomial, and once the choice has been made, the remaining roots will be determined. So far, as in the case of the parameter space, the designer has the freedom of choosing several of the roots of the characteristic equation, but is unable to determine whether or not the remaining roots will help in design, or even make the system stable. This is a tedious trial and error process, and it is very likely that for the chosen roots there will not be a satisfactory set of remaining roots. There are some useful methods like the Polak method and the Ross-Warren method, employing filter compensation, but they are essentially one-variable parameter methods and are not applicable in a general sense.

4. CONSTANT ζ SURFACES

It was shown in Section 3 that the locus on the α β γ space for given values of ζ and ω_n was the line defined by the intersection of the two planes represented by equations 3.5 which are:

$$\alpha B_1 + \beta C_1 + \gamma D_1 + E_1 = 0$$

$$\text{and} \qquad \alpha B_2 + \beta C_2 + \gamma D_2 + E_2 = 0 \qquad 3.5$$

The above statement can be interpreted also as "for a given ζ, the constant ω_n loci on the α β γ space are straight lines", so that by keeping ζ constant for each value of ω_n there will be a set of two planes in the form of equations 3.5 with the intercept a straight line in the three dimensional space α β γ. By projecting this straight line onto one of the planes generated by the coordinate axes we will be able to summarize all the information pertaining to this straight line in one plane. Figure 4.1 shows the intersection of two planes ABC and DEJ as the line GH and its extension (dotted line). The projection of GH on the plane is the line HF and its extension. As can be seen, equal increments in γ will produce equal advances along the projection. Therefore, linear interpolation can be done along the projection line HF, and the information concerning the line of intersection can be summarized in the α β plane. The line HF and its extension can easily be determined by finding the points H and F. Setting $\gamma = 0$ in equations 3.5, and solving for α and β we obtain the point

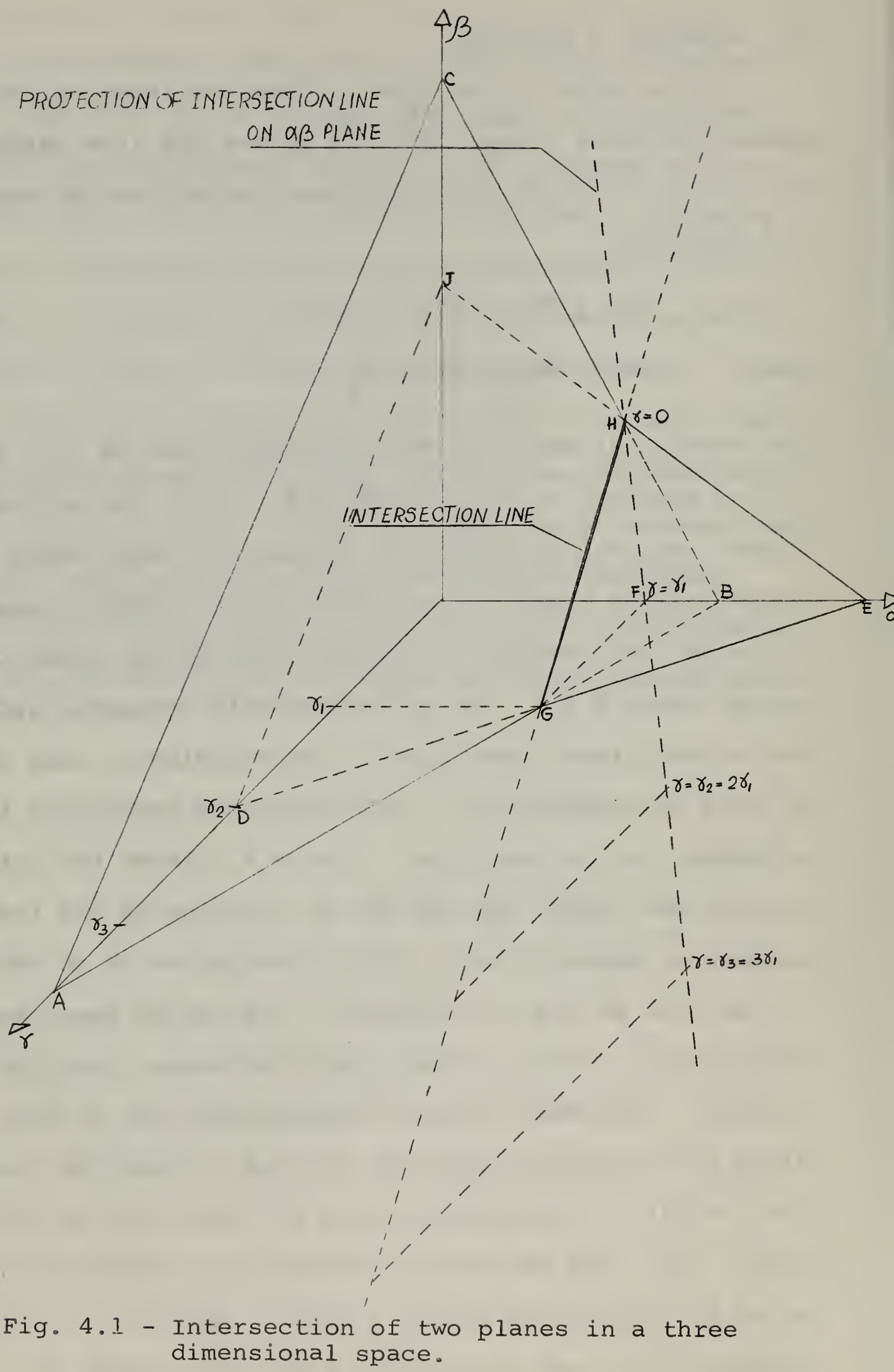


Fig. 4.1 - Intersection of two planes in a three dimensional space.

H, applying Cramer's rule:

$$\alpha H = \frac{(-E_1C_2 + E_2C_1)}{\Delta H}$$

4.1

and $$\beta H = \frac{(-B_1E_2 + B_2E_1)}{\Delta H}$$

where $$\Delta H = B_1C_2 - B_2C_1$$

Similarly for point F, setting $\beta = 0$

$$\alpha F = \frac{-E_1D_2 + E_2D_1}{\Delta F} \qquad 4.3$$

and $$\beta F = 0.0$$

It is possible also, to determine a "measuring-stick" along the line HF by determining the value of γ, given by

$$\gamma_1 = \frac{(-B_1E_2 + B_2E_1)}{\Delta F} \qquad 4.4$$

where $$\Delta F = B_1D_2 = B_2D_1 \qquad 4.5$$

As can be seen, for an increment $\Delta\gamma = \gamma_1$, we must advance along the projection line equal distances HF. We can draw the projection line in the α β plane and by using linear interpolation, label along this line the values of γ. Figure 4.2 illustrates the construction method used in the two planes shown in the three-dimensional picture in Figure 4.1. For this we have drawn each one of the planes represented in equation 3.5 by the method proposed in Section 2. The intersection of the two planes in the α β plane for $\gamma = 0$ is the point H and the intersection in the α β plane through $\gamma = \gamma_1$, is the point F. Therefore, the

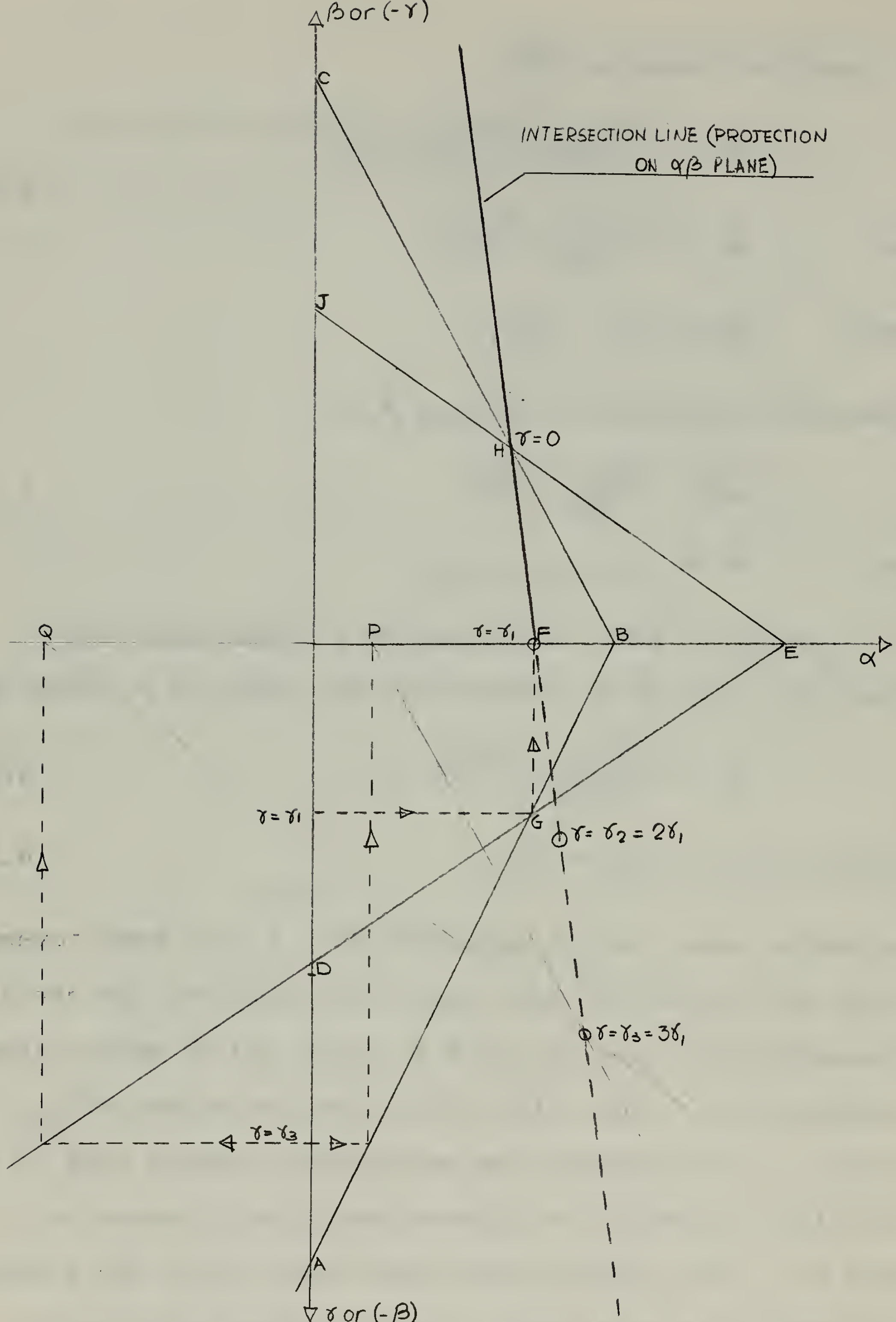


Fig. 4.2 - Two dimensional representation of the intersection of the two planes shown in Fig. 4.1.

projection line is HF. Any other point along this line can be determined by: a) picking up the corresponding value of γ_1 (see γ_3) and projecting this value onto the α axis through the lines on the α γ plane (ED and BA in this case). These projections will determine the points P and Q on the α axis. By drawing a line parallel to BC through P and a line parallel to EJ through Q we determine the point $\gamma = \gamma_3 = 3\gamma_1$, along the line HF, from which we can obtain the corresponding values of α and β. b) Linear interpolation can be used also, by laying off the distance HF along the projection of the line for equal increments, $\Delta\gamma = \gamma_1$. The construction method was used only as a graphical help to understand the linear characteristics and the possibility of linear interpolation. Everything can be done analytically. The equation of the projection line can be obtained by using the formula for a straight line, given two points H and F. An example follows that will illustrate the method.

<u>Example 4.1</u>: Assume the system shown in figure 4.3 is to have a dominant pair of poles with $\zeta = 0.5$ and $\omega_n = 2.0$. The characteristic equation is

$$(S + 0.5)(S + 1)(S + 5)(S + 10) + K\,K_a S^2 + K\,K_t S + K\,K_1 \tag{4.6}$$

By rearranging terms we get:

$$S^4 + 16.5S^3 + (K\,K_a + 73)S^2 + (K\,K_t + 82.5)S + (K\,K_1 + 25) = 0 \tag{4.7}$$

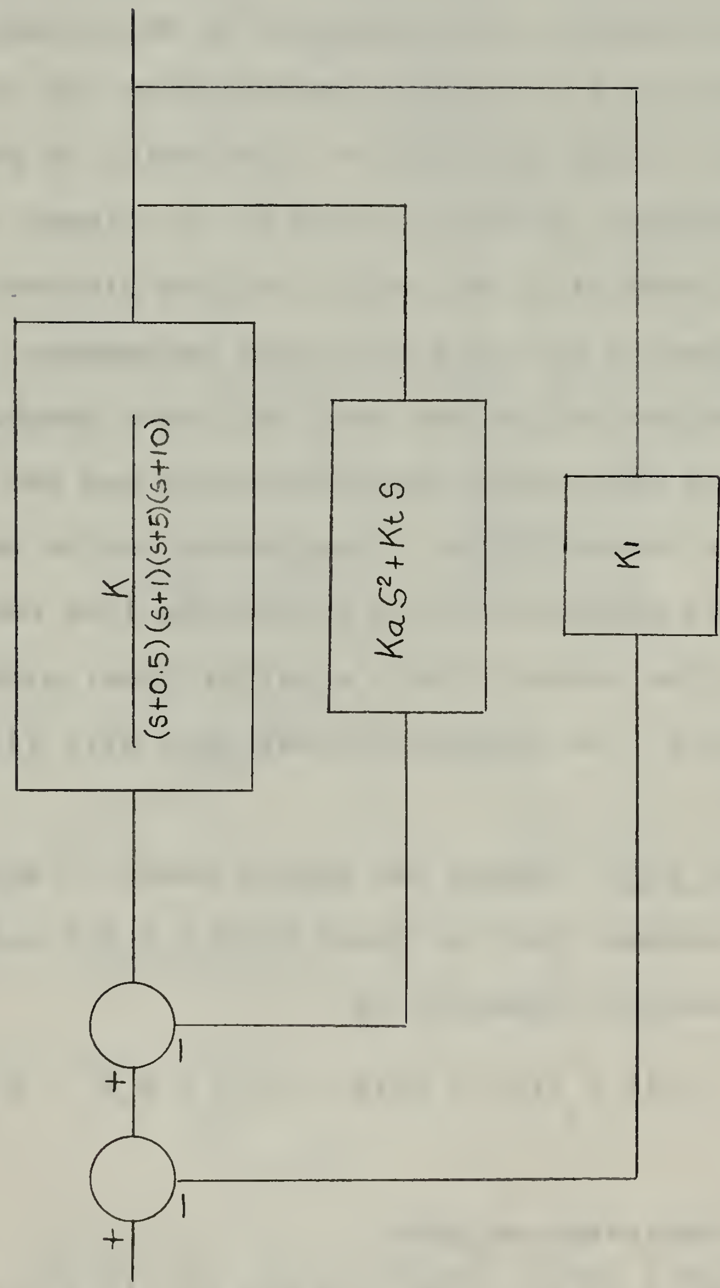


Fig. 4.3 - System with three variable parameters,
Example 4.1.

or

$$S^4 + 16.5S^3+(\alpha + 73)S^2+(\beta + 82.5)S + (\gamma + 25) = 0$$

where $\alpha = K K_a$ $\beta = K K_t$ $\gamma = K K_1$

by setting $\zeta = 0.5$ and $\omega_n = 2.0$ we get the equations:

$$4\alpha - \gamma + 135 = 0$$

and $4\alpha - 2\beta + 110 = 0$ 4.8

by making $\gamma = 0$ we get $\alpha H = -33.75$

and $\beta H = -12.0$ 4.9

For $\beta = 0$ we get $\alpha F = -27.75$ 4.10

and $\gamma = 24.0$

We can now draw the straight line on the α β plane as shown in figure 4.4.

By inspection of equation 4.7 we can see that one of the roots will go in the right-half side of the s-plane if γ is negative and of absolute value greater than 25; therefore, for the system to be stable γ must be $\gamma > -25$. By setting $\gamma = 0$ the following roots are obtained for equation 4.6:

$S_{1,2} = -1.0 \pm 1.73j$ (desired value of roots)

$S_3 = -14.055$

$S_4 = -0.446$

The corresponding values of α and β are -33.75 and -12.0 respectively. Figure 4.5 shows the location of the roots for several values of γ. Two important features are

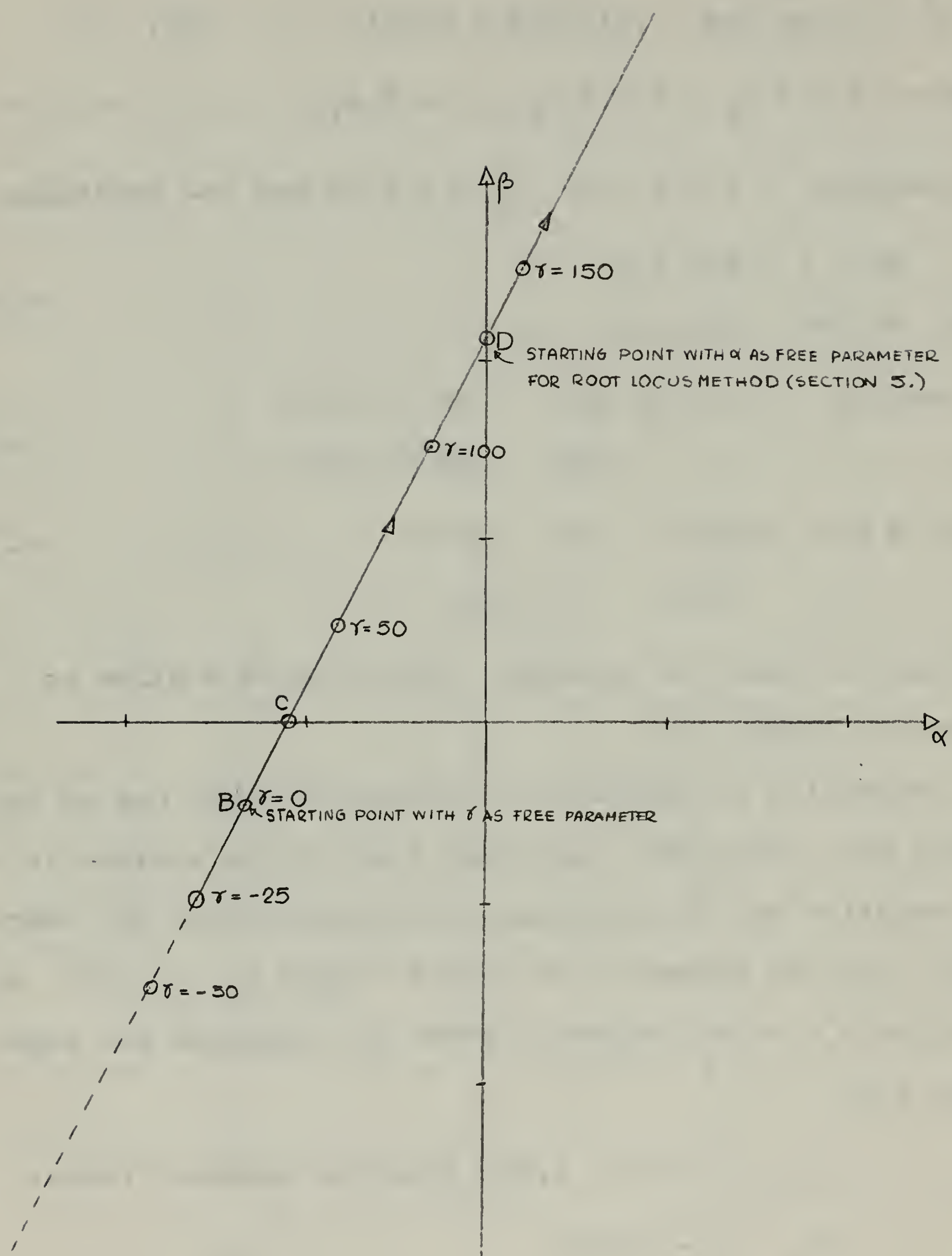


Fig. 4.4 - Straight line representation of ω_n line projection on the $\alpha\beta$ plane - Example 4.1.

noticeable on this graph: a) Due to the relationships established in equations 4.8, the desired pair of complex-conjugate roots remains in the same place on the s-plane for all values of γ. b) The remaining roots of the polynomial travel through the s-plane in the same fashion as on the root-locus. This is essentially a method for determining the loci of the roots when there is only one variable parameter in the system. This fact is justified because two degrees of freedom have been used in fixing two of the roots of the polynomial. Therefore it is possible to obtain a root-locus for the remaining roots of the polynomial. The method is outlined in Section 5 and proves to be useful in design for systems with several variable parameters. Table 4.1 gives the values of the remaining roots and the corresponding values of α, β and γ for the characteristic equation of Example 4.1.

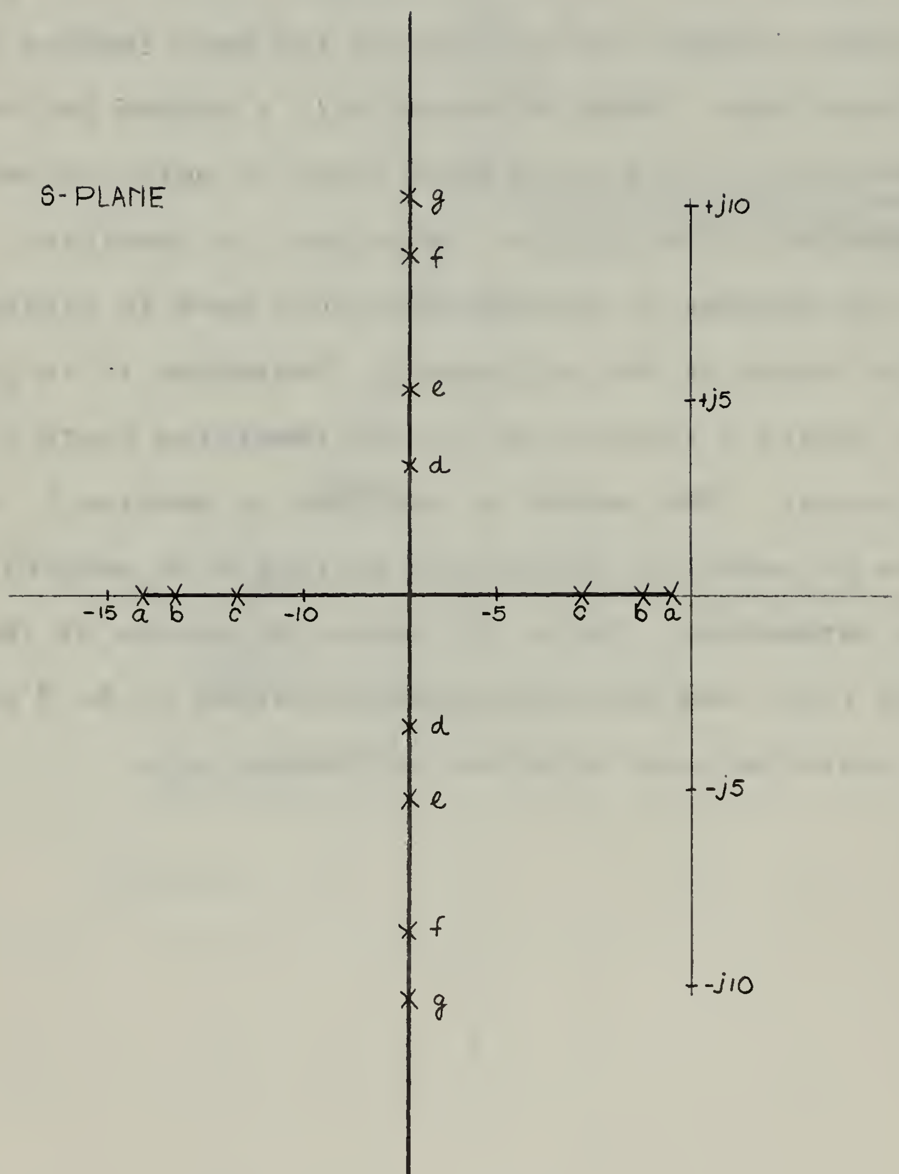


Fig. 4.5 - Locus of remaining poles in characteristic equation of system in Example 4.1.

No.	γ	S_3	S_4	α	β
a	0	-14.055	-0.446	-33.75	-12.0
b	39	-13.29	-1.21	-24	7.0
c	103	-11.77	-2.73	- 8	39
d	231	-7.25+3.37j	-7.25-3.37j	24	103
e	295	-7.25+5.22j	-7.25-5.22j	40	135
f	495	-7.25+8.79j	-7.25-8.79j	90	235
g	625	-7.25+10.3j	-7.25-10.3j	122.5	300
h	1007	-7.25+45.3j	-7.25-45.3j	218	491

Table 4.1 - Corresponding values of S_3, S_4, α and β for system of Example 4.1.

The straight-line property of the constant ω_n lines can also be used for interpolation. In order to illustrate this let us assume that computer results[3] yields curves $A(\zeta = a, \gamma = \gamma_1)$ and $B(\zeta = a, \gamma = \gamma_2)$ on figure 4.6; but because of some design considerations the range of admissible values for α and β is the region OPQR on the α β plane. It is possible then to get a fairly accurate picture of the shape of the curves in that region by drawing straight lines between points on curves A and B having corresponding values of ω_n, and by linear interpolation on these lines to obtain curves for intermediate values of γ between γ_1 and γ_2. Curve C corresponds to $\gamma = (\gamma_1 + \gamma_2)/2$ and was obtained by dividing the straight lines into two equal segments. As it can be seen the loop on curve A tends to disappear as we approach curve B.

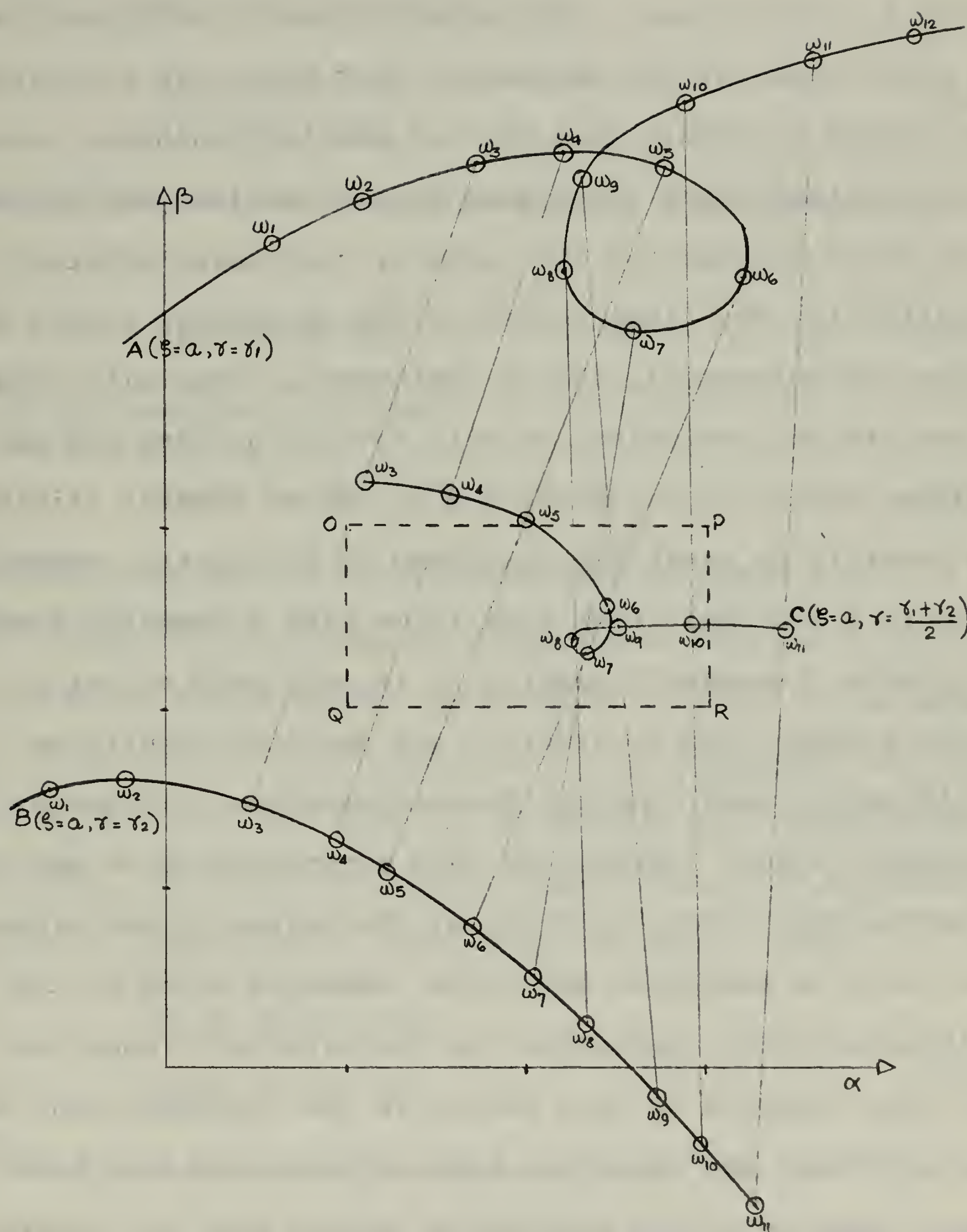


Fig. 4.6 - Use of linear interpolation in constant ζ surface using constant ω_n lines.

5. MODIFIED ROOT LOCUS

There are many instances where the performance specifications for a system require a pair of complex conjugate roots for the system. The parameter-space techniques tell us what values of the parameters α, β and γ are suitable for the system to obtain this pair of complex conjugate roots. Unfortunately these techniques do not disclose any information about the rest of the roots of the characteristic equation for the given values of the parameters unless we solve the polynomial, and if dominance or stability conditions are not satisfied, we will have to go back and make another choice of values of α β γ. After several trials it is possible to establish the trend of the roots. However, this is a difficult task even if we have a computer program available, because in general it is very hard to get a clear picture, and in order to get detailed results we would need a very complete three-dimensional picture of the parameter space. Because of this inconvenience we may use a method that gives a picture of the system at one glance and tells us something about the remaining roots of the polynomial which represents the characteristic equation. The name assigned to this method is the "modified root locus", because the resulting locus follows the same techniques applied to the root locus, except that the desired pair of roots remains in the same place on the s-plane while the variable parameters α, β and γ are changed.

Let us assume we have a system with characteristic equation $G(S, \alpha, \beta, \gamma)$, and a pair of complex conjugate roots with given values of ζ and ω_n is desired. It is possible to reduce the order of the characteristic equation by making the division:

$$\frac{G(S, \alpha, \beta, \gamma)}{S^2 + 2\zeta\omega_n + \omega_n^2} \qquad 5.1$$

The quotient of this division will be a polynomial of order n-2:

$$F(S, \alpha, \beta, \gamma) \qquad 5.2$$

and the remainder will be

$$F_1(\alpha, \beta, \gamma)S + F_2(\alpha, \beta, \gamma) \qquad 5.3$$

The division has to be an exact one, in order for $S^2 + 2\zeta\omega_n + \omega_n^2$ to satisfy the equation $G(S, \alpha, \beta, \gamma) = 0$. Therefore, the remainder (Expression 5.3) has to be equal to zero, and for this condition to be satisfied, the values of $F_1(\alpha, \beta, \gamma)$ and $F_2(\alpha, \beta, \gamma)$ have to go independently to zero. This can be readily seen because for any arbitrary value of S equation 5.3 has to go to zero. Then:

$$F_1(\alpha, \beta, \gamma) = 0$$

$$\text{and} \quad F_2(\alpha, \beta, \gamma) = 0 \qquad 5.4$$

This set of equations is, in some cases, the very same set of equations represented in equation 3.5,or a linear combination of them. Therefore, they represent a line on the three-dimensional space $\alpha\ \beta\ \gamma$. We can apply the same techniques used in Section 4 to obtain a graphic or mathematic representation and obtain the corresponding set of values of α, β and γ that will yield a pair of complex roots having values ζ and ω_n. But what about the remaining roots of the polynomial? Will they yield the desired dominance conditions? Will the system be stable? What are the required values of α, β and γ? Will these values be physically attainable? The answer to these questions lies in the quotient of the division represented by Expression 5.2. The fact that two of the roots have been fixed means that we have implicitly used two degrees of freedom. Therefore, there is left only one degree of freedom which is the essence of the root-locus method. Its techniques can be applied using the following procedure:

From equations 5.4, we can get any two parameters as a function of the third parameter. Let us assume we are to solve α and β as functions of γ; then equations 5.4 yield:

$$\alpha = G_1(\gamma)$$

and $$\beta = G_2(\gamma),$$

or, solving for α and γ:

$$\alpha = G_1^{1}(\beta)$$

and $$\gamma = G_2^{1}(\beta), \qquad 5.5$$

or, solving for β and γ:

$$\beta = G_1^{11}(\alpha)$$

and $$\gamma = G_2^{11}(\alpha).$$

Now by substituting the values of say, α and β in Expression 5.2 (the quotient) we obtain

$$F(S, \gamma)$$

or if we use α and γ:

$$F(S, \beta) \qquad 5.6$$

or if we use β and γ:

$$F(s, \alpha)$$

Now we are ready to apply the root-locus techniques to the polynomial F(S). Since this polynomial has also to satisfy the characteristic equation we can say that:

$$F(S, \gamma) = 0 \qquad 5.7$$

by rearranging terms of this polynomial we get

$$\frac{\gamma N(S)}{D(S)} = -1 \qquad 5.8$$

The root locus for γ will be the locus of the remaining roots of the characteristic equation. From this root locus it is possible to establish stability and dominance conditions. Choose the more convenient value of γ, and by substituting it into equation 5.5 get the corresponding values of α and β; or by a simple look at the straight-line method established in Section 4, obtain the physically attainable values of α and β. This modified root locus will also allow the designer to select the best choice of the remaining roots of the polynomial that will help to attain other performance specifications, such as bandwidth and gain.

The method described above for three parameters can be generalized for dynamic systems with k variable parameters. In this case the designer has (k-1) choices of the roots and there will be one "free parameter" that can be varied at will for application of the root-locus techniques. The characteristic equation will be divided:

$$\frac{G(S,\ \alpha,\ \beta,\ \gamma,\ \ldots\ldots\ k)}{S^{k-1} + b_{k-2}S^{k-2} + \cdots\cdots\ b_o} \qquad 5.9$$

where the denominator of equation 5.9 corresponds to the product of the chosen (k-1) roots.

The quotient of this division will be a polynomial of order n - k + 1

$$F(S,\ \alpha,\ \beta,\ \gamma\ \ldots.\ k) \qquad 5.10$$

and the remainder will be

$$F_1(\alpha,\beta,\gamma,\ldots.k)S^{k-2} + \ldots\ldots\ F_{k-2}(\alpha,\beta,\gamma\ldots.k)S + F_{k-1}(\alpha,\beta,\gamma\ldots) \qquad 5.11$$

Again, if the division is going to be satisfied with the remainder equal to zero a set of k - 1 simultaneous equations are obtained.

$$\begin{aligned} F_1(\alpha,\ \beta,\ \gamma\ \ldots\ k) &= 0 \\ F_2(\alpha,\ \beta,\ \gamma\ \ldots\ k) &= 0 \\ &\vdots \\ F_{k-2}(\alpha,\ \beta,\ \gamma\ \ldots\ k) &= 0 \\ F_{k-1}(\alpha,\ \beta,\ \gamma\ \ldots\ k) &= 0 \end{aligned} \qquad 5.12$$

From equations 5.12, it is possible to solve for (k-1) parameters as functions of a k^{th} parameter and by substitution in Expression 5.10 we get:

$$F(S, \text{free parameter}) = 0 \qquad 5.13$$

The application of root locus techniques to equation 5.13 will produce the locus of the remaining roots of the characteristic equation. These techniques are useful in various cases, depending on the applications:

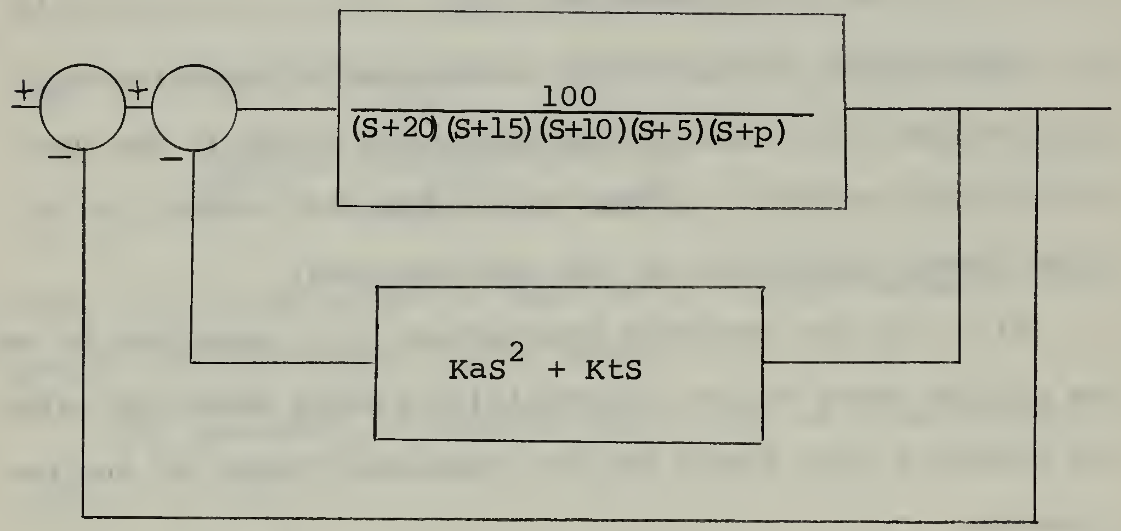
a). for two variable parameters it is possible to set one of the roots of the polynomial at a large negative value and obtain a root locus for the remaining roots of the polynomial.

b). for three parameters the choice of the dominant pair of complex conjugate roots can be made, allowing determination of the remaining roots of the polynomial. It is possible to give two of the real roots large negative values and determine the remaining roots from the root locus, choose the ones that help us to get as close as possible to the desired performance specifications. Similar considerations can be made for k variable parameters.

The calculations involved are simple and provide an overall picture of the system and its possibilities. In many cases the quotient is a function of only one of the parameters; therefore, the equation for the root locus is available without the need of substitutions (see Problem 5.2). With a computer program for root locus the job is less laborious.

Several examples are available in order to clarify the procedure and the techniques. Each one of the cases outlined above is studied.

<u>Example 5.1</u>:



System with three variable parameters - Example 5.1.

Fig. 5.1

The system on Figure 5.1 is to be compensated using tachometer feedback and accelerometer feedback. We wish to insure dominance of a pair of complex conjugate roots with $\zeta = 0.5$ and $\omega_n = 2.0$. We wish also to know what are the permissible values of the remaining roots of the characteristic equations and, if it is possible, to obtain dominance conditions the required values of p, k_a and k_t. The transfer function for the system on Figure 5.1 is:

$$F(S) = \frac{100}{S^5+(\alpha+50)S^4+(50\alpha+830)S^3+(830\alpha+100\beta+6250)S^2+(6250\alpha+100\gamma+1500)S+(15000\alpha+25)} \qquad 5.9$$

where $p = \alpha$, $K_a = \beta$, and $K_t = \gamma$

By dividing by $S^2 + 2S + 4$ we get the quotient:

$$S^3 + (\alpha+48)S^2 + (48\alpha + 730)S + (730\alpha + 100\beta + 4598) \quad 5.10$$

and the remainder

$$(4598\alpha+100\gamma - 200\beta+2884)S + 12080\alpha - 400\beta - 18292 \quad 5.11$$

In order for the remainder to be zero for arbitrary values of S, it is necessary that

$$\begin{aligned} 4598\alpha + 100\gamma - 200\beta + 2884 &= 0 \\ \text{and} \quad 1208\alpha - 400\beta - 18292 &= 0 \end{aligned} \quad 5.12$$

So far, two degrees of freedom have been used; therefore we are left with only one of the parameters variable (free parameter). The other two are dependent functions of this variable parameter (equations 5.12).

It is easiest to have α as the variable parameter, because most of the coefficients in the quotient, Expression 5.11, are functions of α, and also because β is easily expressed as a function of α, in the second of equations 5.12. From these considerations:

$$\beta = \frac{3020\alpha - 4573}{100} \quad 5.13$$

By substituting the value of β in Expression 5.10 and making it equal to zero, we get:

$$S^3 + (\alpha+48)S^2 + (48+730)S + (3750\alpha + 25) = 0 \quad 5.14$$

Now, by applying the root locus technique,

$$\frac{\alpha(S^2+48S+3750)}{S^3+48S^2+730S+25} = -1$$

$$\text{or} \quad \frac{\alpha(S + 24 - 56.338j)(S + 24 + 56.338j)}{(S+23.847-j12.377)(S+23.847+j12.377)(S+0.0343)} = -1$$

The root locus for the variable parameter α is shown in figure 5.2. For any point on the given root locus it is possible to determine the required value of α and the corresponding values of β and γ are determined from the relations expressed in equation 5.12. The graphical method may also be used. The root locus in figure 5.2 shows all possible values of the remaining three roots of the polynomial. For low values of α, the real root will be very close to the pair of complex poles that are desired as dominant. For high values of α, the real root will go far in the left of the s-plane and the other pair of complex roots will be very close to the generating zero of the root locus. As can be seen, the system allows in this case an improvement in the bandwidth by choosing the value of α that will make the real root lie in the far left of the s-plane and selection of the complex roots to insure a dominance condition (real part of the complex roots about ten times the real part of the dominant roots). Other considerations can be made such as what values of α will yield permissible values of K_2 and K_t, without saturating the amplifiers that will provide these gains, etc.

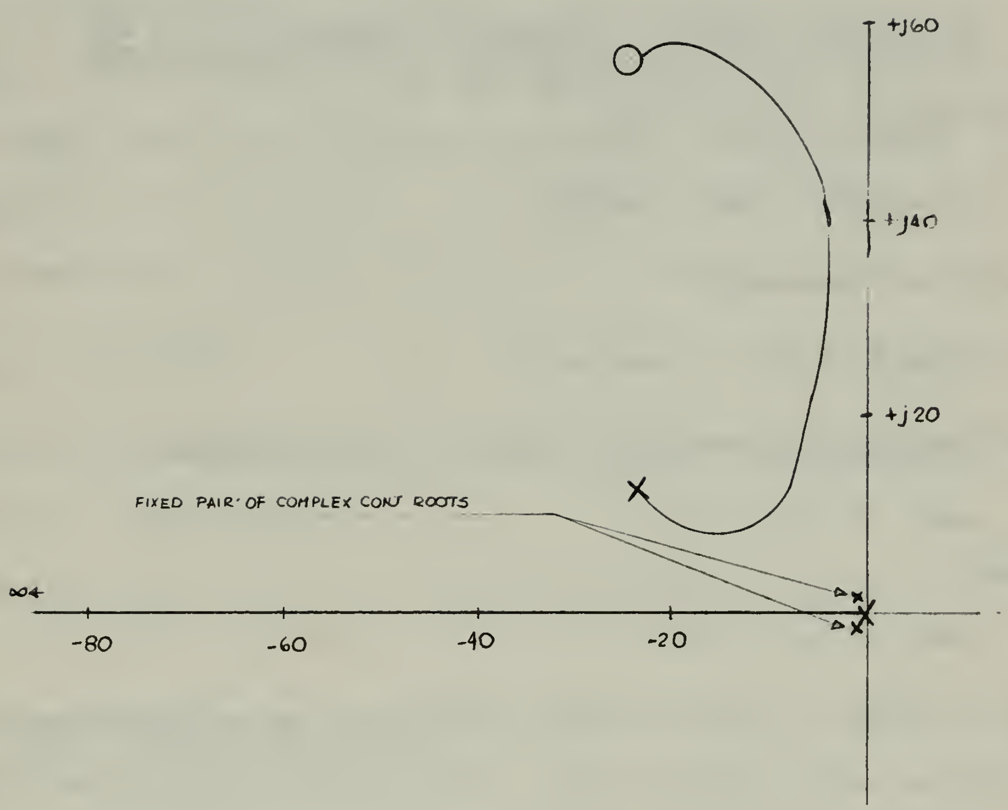


Fig. 5.2 - Modified root locus for remaining roots of characteristic equation for system in Fig. 5.1.

Example 5.2:

A system has the following characteristic equation:

$$G(S) = S^5 + 16S^4 + 40S^3 + (\alpha+10)S^2 + (\beta+50)S + (\gamma+25)$$

It is desired to have a pair of complex conjugate roots with $\zeta = 0.5$ and $\omega_n = 2.0$. By making the division:

$$\frac{S^5 + 16S^4 + 40S^3 + (\alpha+10)S^2 + (\beta+50)S + \gamma + 25}{S^2 + 2S + 4} \qquad 5.15$$

we obtain the quotient:

$$S^3 + 14S^2 + 8S + (\alpha-62) \qquad 5.16$$

and the remainder

$$(-2\alpha+\beta+142)S + (-4\alpha+\gamma+273) = 0 \qquad 5.17$$

By making the remainder equal to zero we get:

$$-2\alpha + \beta + 142 = 0$$

and

$$-4\alpha + \gamma + 273 = 0 \qquad 5.18$$

As is easy to see in this case there is no need for substitution, since the quotient is a function only of α. However, the values of β and γ will depend on the value chosen for α, from the relationships expressed in equations 5.18.

Rearranging Expression 5.16 for root locus application:

$$S^3 + 14S^2 + 8S + (\alpha-62) = 0 \qquad 5.19$$

or

$$\frac{\alpha}{S^3 + 14S^2 + 8S - 62} = -1 \qquad 5.20$$

or

$$\frac{\alpha}{(S - 1.74646)(S + 2.7266)(S + 13.01977)} = -1 \qquad 5.21$$

The root locus for equation 5.21 is shown in figure 5.3. As can be seen it is not possible to obtain dominance conditions. The system will become unstable for very large or very small values of α. In this case, with very few steps it was possible to obtain a clear picture of the system. It is obvious that the root locus must have the same shape whatever parameter is chosen. We can check this fact by choosing say, γ as our variable parameter. From equations 5.18 the expression for α is:

$$\alpha = \frac{\gamma + 273}{4} \qquad 5.22$$

By substitution of the expression for α in the quotient (Equation 5.19) we get

$$4S^3 + 56S^2 + 32S + \gamma + 273 - 248 = 0$$

or

$$\frac{\gamma}{4(S^3 + 14S^2 + 8S + 6.25} = - 1$$

If $K = \gamma/4$, we get:

$$\frac{K}{(S^3 + 14S^2 + 8S + 6.25)} = - 1$$

or

$$\frac{K}{(S + 13.439)(S + 0.28 \pm j0.622)} = - 1 \qquad 5.23$$

The root locus for equation 5.23 is shown in figure 5.4 and it is exactly the same as the one on figure 5.3. The part which is not present is that corresponding to negative values of γ. Had we drawn the root locus for negative values of γ this part would appear. The points where the

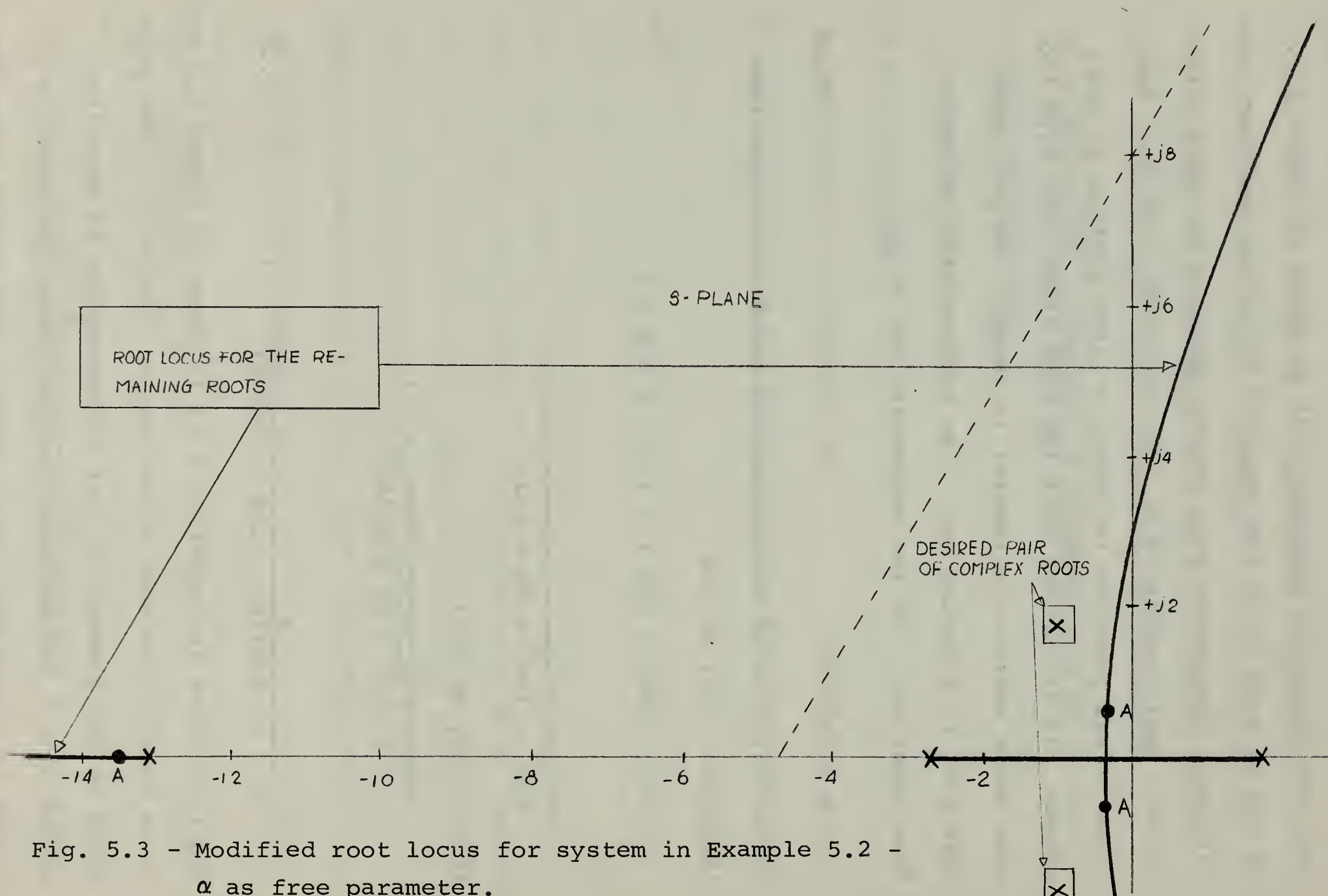
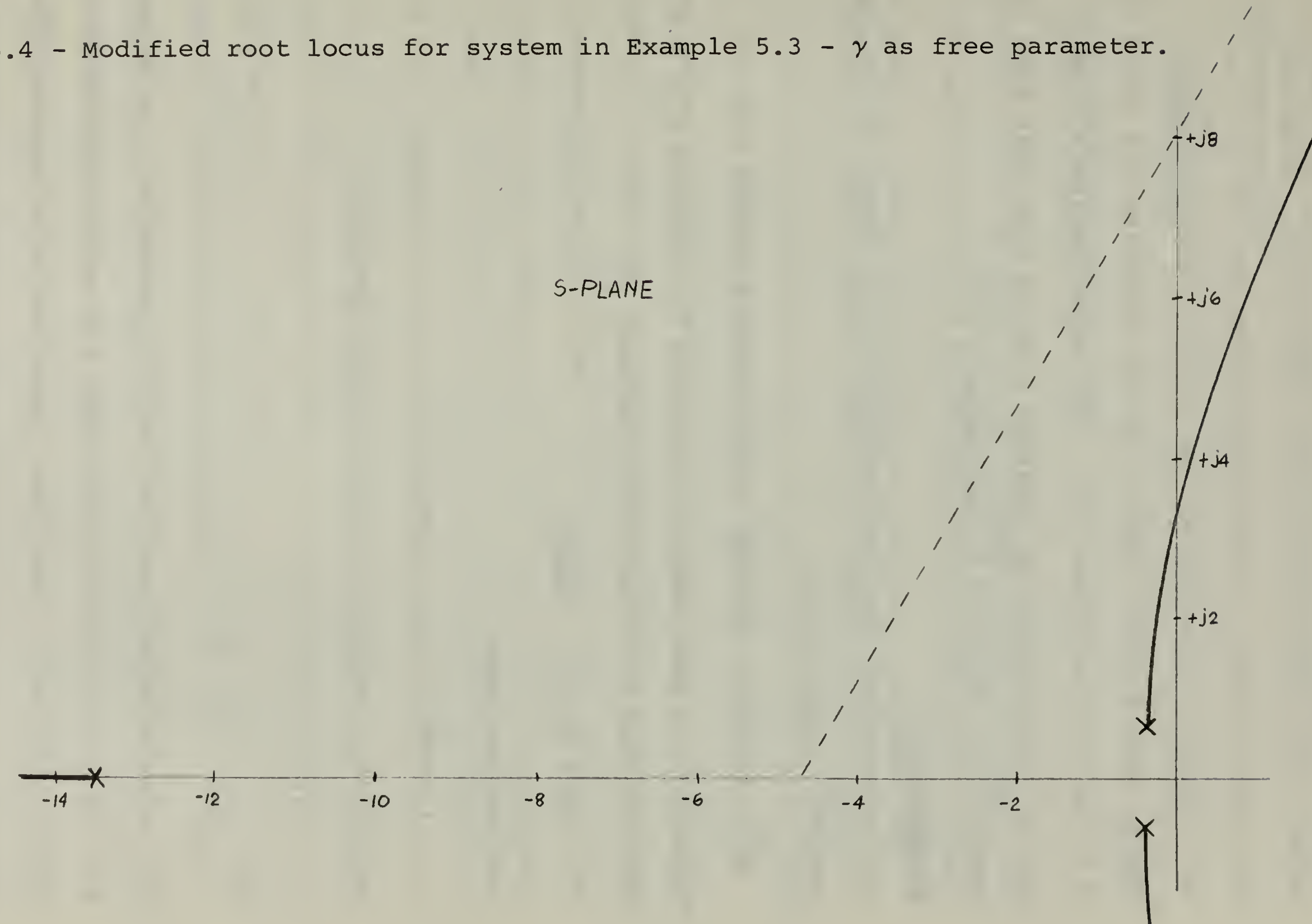


Fig. 5.3 - Modified root locus for system in Example 5.2 - α as free parameter.

Fig. 5.4 - Modified root locus for system in Example 5.3 - γ as free parameter.



generating poles for the root locus of α appear are the points for which the value of γ make $\alpha = 0$. Similarly the points where the generating poles for the root locus of γ appear correspond to the value of α that makes $\gamma = 0$. We can make a quick check. Let us determine the value of α at point A on figure 5.3. This value yields $\alpha = 68.25$, and by substituting in equation 5.22 we find that this is exactly the value of α that makes $\gamma = 0$.

<u>Example 5.3</u>:

Let us consider again the example 4.1 in which the characteristic equation is

$$S^4 + 16.5S^3 + (\alpha+73)S^2 + (\beta+82.5)S + (\gamma+25) = 0$$

by using the same desired values of $\zeta = 0.5$ and $\omega_n = 2.0$ and dividing by S^2+2S+4, the quotient is

$$S^2 + 14.5S + (\alpha+40)$$

and the remainder is

$$(-2\alpha+\beta-55.5)S + (-4\alpha+\gamma-135)$$

Setting the remainder equal to zero, we have

$$-4\alpha + \gamma - 135 = 0$$

and

$$-2\alpha + \beta - 55.5 = 0 \qquad 5.24$$

By comparing equations 5.24 with equations 4.8, in example 4.1, it is easy to see that these two sets of equations are exactly the same; therefore, equations 5.24 represent the straight line that defines a constant ω_n on the constant ζ

surfaces. By making a root locus set up with α as a free parameter we get:

$$\begin{aligned} \frac{\alpha}{(S^2+14.5S+40)} &= -1 \\ \frac{\alpha}{(S+10.995)(S+3.705)} &= -1 \end{aligned} \tag{5.25}$$

The root locus is shown as the slashed lines in figure 5.5. Poles marked D are the generating poles using α as a free parameter. Had γ been chosen as the free parameter the generating poles would be the ones marked as B on figure 5.5, and the root locus using γ as the free parameter would be the part shown as heavy lines, plus the slashed lines. For roots at points between B and D the value of α would be negative. This fact can be checked in figure 4.4 where the corresponding points B and D are marked. The point of importance in this case is that since root locus techniques consider only positive values of the variable parameter, for each parameter chosen as free parameters, we will get only part of the overall root locus. Therefore, if the results with a positive value of the free parameter are not satisfactory, it is good practice to make a quick check for negative values of the free parameter. This can be done easily by starting at the generating poles and going in the opposite direction. For this specific example, we can see that for negative values of α, the root locus will go from D towards B (see Figure 5.3), and eventually one of the roots will cross the $j\omega$ axis and the system will become unstable.

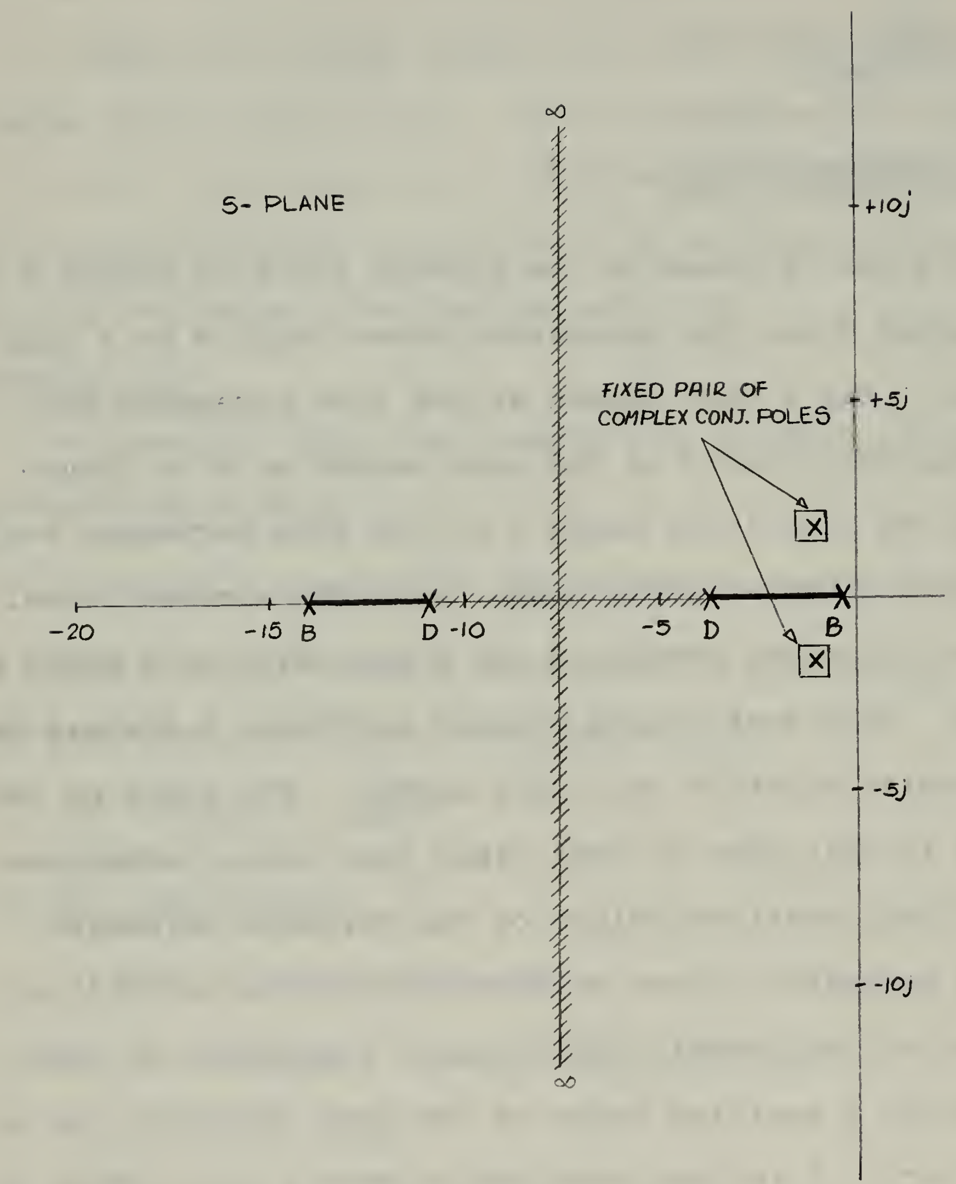


Fig. 5.5 - Modified root locus for Example 5.3.

Example 5.4- Example with parameters product.

There are some cases where the characteristic equation contains products of parameters. This technique is applicable only if the quotient of the division is a function of only one of the parameters. Let us assume a system with characteristic equation:

$$S^4 + (\alpha+15)S^3 + (20\alpha+200)S^2 + (\gamma+600)S + \alpha\beta + 6000 \qquad 5.26$$

and the system is required to have a pair of complex conjugate roots with $\zeta = 0.5$ and $\omega_n = 10$. From the division we get quotient

$$S^2 + (\alpha+5)S + (10\alpha+50) \qquad 5.27$$

and the remainder

$$(-200\alpha+\gamma-400)S + (\alpha\beta-1000\alpha+1000) = 0 \qquad 5.28$$

The root locus technique can be applied to the Expression 5.27, thus,

$$\frac{\alpha(S+10)}{(S^2+5S+50)} = -1$$

or

$$\frac{\alpha(S+10)}{(S+2.5-6.644j)(S+2.5+6.644j)} = -1$$

and the root locus can be seen in figure 5.6

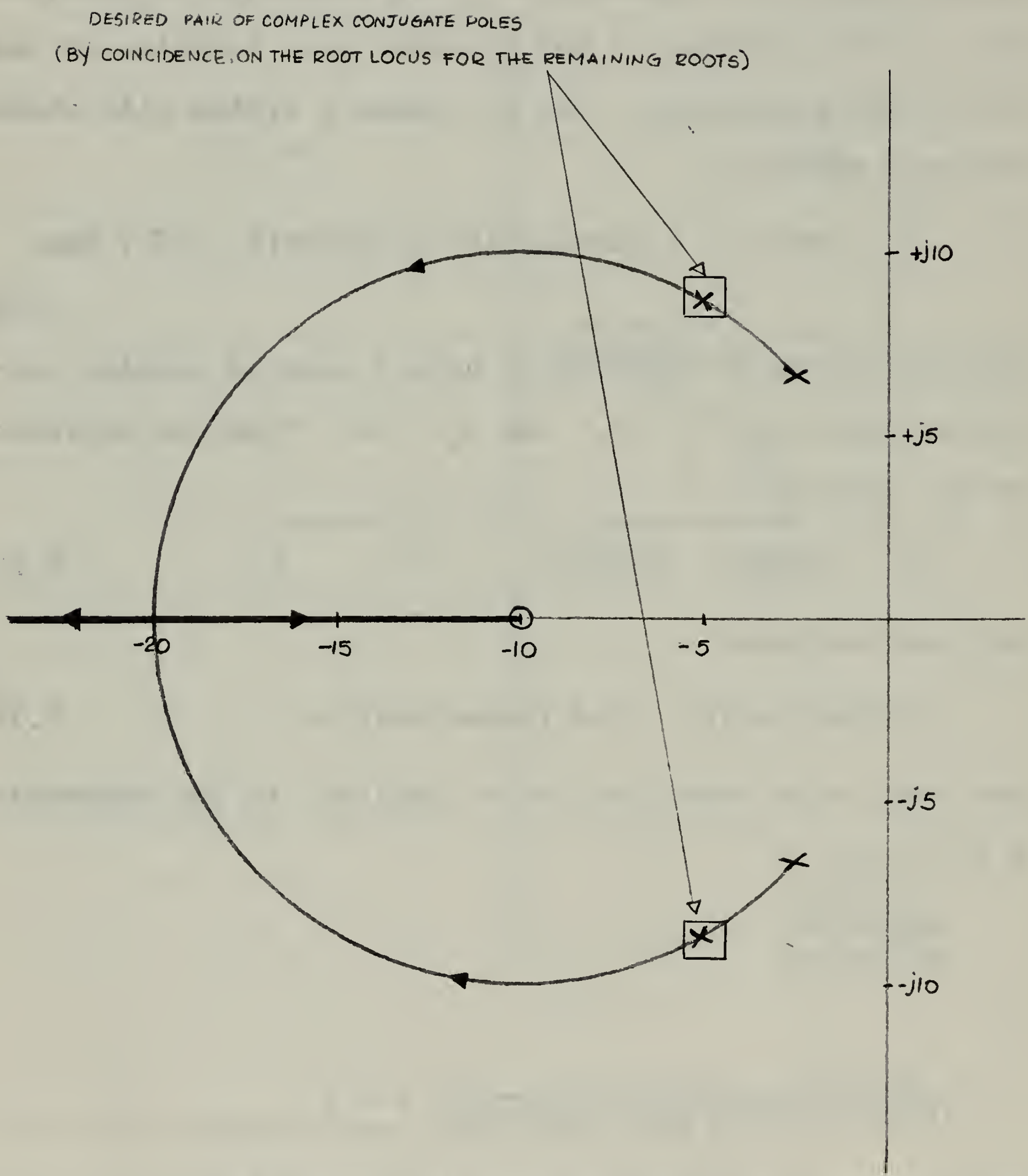


Fig. 5.6 - Modified root locus for Example 5.5 - α as free parameter.

The choice for dominance can be made as a pair of poles at S = -20. From the root locus we get

$$\alpha = 35$$

and from Expression 5.28:

$$\beta = \frac{1000\alpha - 1000}{\alpha}$$

$$\beta = \frac{34000}{35} = 971$$

and

$$\gamma = 200\alpha + 400 = 7400$$

we could also obtain the values by multiplication.

$$G(S) + (S+20)^2(S^2+10S+100) = 0$$

$$G(S) = S^4+50S^3 + 900S^2 + 8000S+40.000 = 0 \qquad 5.29$$

By comparison of the two expressions for G(S) (Eqs. 5.26 and 5.29) we get

$$\alpha + 15 = 50$$

or $\qquad : \quad \alpha = 35$

$$20\alpha + 200 = 900$$

and $\quad \gamma + 600 = 8000 \quad : \quad \gamma = 7,400$

and $\quad \alpha\beta + 6000 = 40.000 \quad : \quad \beta = \frac{34000}{35} = 971$

which checks with the results above.

This is a special case where the quotient (Eq. 5.27) is a function of only one of the parameters. In other cases the root locus technique is not applicable because the free parameter appears in the quotient as K, K^2, K^3, etc., depending on the complexity of the parameter products in the characteristic equation. However, there exists the possibility that the designer has enough freedom to modify the system configuration and make the selection of the variable parameters in such a way that no products appear, or in case of products, that the quotient be a function of only one parameter. All this can be achieved by changing some feedback paths or series compensation schemes.

6. SINGULAR LINES AND SELF-ADAPTIVE SYSTEMS

In the analysis of systems with two variable parameters by application of the transformations proposed by Mitrovic (coefficient plane) and Siljak (parameter plane), the presence of systems with what is known as singular lines was detected. A summary of the procedure that led to the detection of this type of system follows. This leads to a better understanding of what is meant by singular lines. From Section 1, the characteristic equation for a given system can be written as:

$$F(S) = a_n S^n + a_{n-1} S^{n-1} + \cdots \quad + a_1 S + a_o = 0 \tag{6.1}$$

Repeating the manipulations of the above equation outlined in Section 1, the following expressions are obtained

$$\begin{aligned} &\sum_{k=0}^{n} (-1)^k a_k \omega_n^{\,k}\, U_{k-1}(\zeta) = 0 \\ &\sum_{k=0}^{n} (-1)^k a_k \omega_n^{\,k}\, U_k(\zeta) = 0 \end{aligned} \tag{6.2}$$

Now by defining the coefficients as

$$a_k = \alpha b_k + \beta c_k + d_k \tag{6.3}$$

and the substitution of a_k in Equation 6.2, this set of equations can be rewritten as the system of linear equations

$$\begin{aligned} \alpha B_1 + \beta C_1 + D_1 &= 0 \\ \alpha B_2 + \beta C_2 + D_2 &= 0 \end{aligned} \tag{6.4}$$

where B_1, B_2, C_1, C_2, D_1 and D_2 are defined as in Equations 1.15. When values of ζ and ω_n are assigned, and the values of B_1, B_2, C_1, C_2, D_1 and D_2 are calculated, each of the equations 6.4 will represent a straight line on the $\alpha\beta$ plane and in a geometric sense, the intersection of these two straight lines will yield the values of α and β that satisfy the system of equations 6.4, and insure that the characteristic equation will have a pair of complex conjugate roots with the assigned values of ζ and ω_n. But the intersection of two lines in a plane is not always a point. Two other cases are possible: a) the two lines are parallel in the $\alpha\beta$ plane and in this case no pair of α, β values will satisfy the system of equations 6.4 (see Fig. 6.1b), or b) the two lines are the same in which case they intercept on an infinite number of points and any point along this line will yield a pair of complex conjugate roots with values ζ and ω_n. The three different cases are shown in Figure 6.1. Figure 6.1c shows the case of coincidence of the two lines; this is known as a singular line case and its name is derived from the matrix algebra manipulation[4] of the system of equations 6.4. The systems with singular line characteristics find application in self-adaptive systems where changes in the dynamic behavior depends on the ability of the system to cope with variations of the parameters. The identification of the values of the parameters and its changes in these self-adaptive systems is done by observation of the output of the system. Identification is a pro-

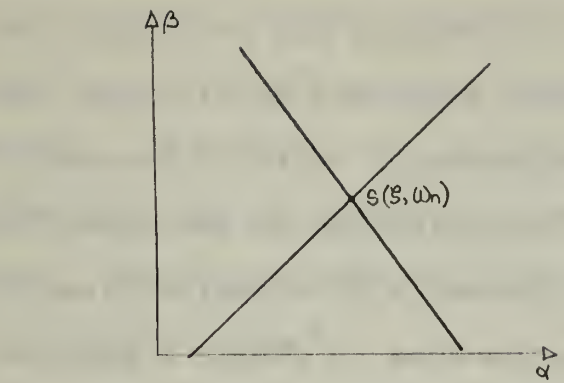


Fig. 6.1a

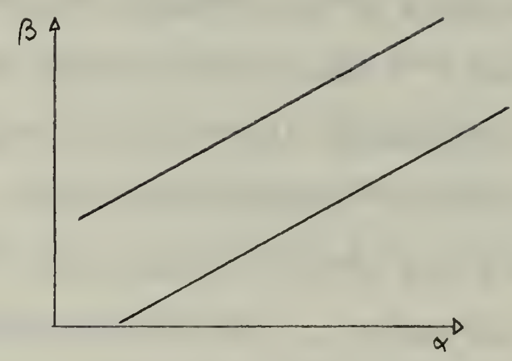


Fig. 6.1b

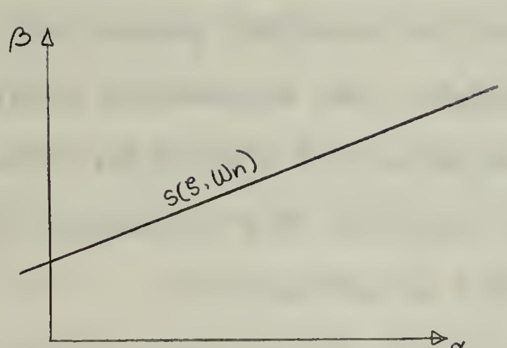


Fig. 6.1c

Different cases of mapping of a point on the s-plane in the $\alpha\beta$ plane.

blem beyond the scope of this work, but literature on the subject is available (see reference 5 and 6). Once the change on the parameters is identified the self-adaptive systems use sensitivity models which are fairly good for very small changes in the parameters. This is because of the approximations made in the selection of the sensitivity models. The idea of using systems with singular lines in a self-sdaptive system came up because it yields a precise mathematical expression (equation of a straight line) that relates the changing parameters. This expression can be simulated by analog or digital means, so the problem of steadyness of the dynamic behavior of the system is solved in part. A sensitivity study on the remaining roots of the polynomial will tell if the desired conditions are going to be jeopardized by the given changes in the parameters and what tolerances can be allowed in the change of the parameters. The attractiveness of the use of systems with singular lines in self-adaptive systems lies in the fact that it is only necessary that one of the parameters be adjusted for changes detected in the other parameter and vice-versa. As was said before, the expression regulating these adjustments is the equation of a straight line, and if sensitivity studies are feasible, the tolerance in the changes of the parameters can be determined.

So far, the detection of systems with singular line characteristics is a difficult job, and even in the case where singular lines are obtained the values of ζ and ω_n

which generate these singular lines might not be satisfactory for the desired performance specifications. The subject is relatively new and research is going on on the matter. Therefore, the possibility of an easy method of detection and set up of singular lines might appear with time. There are computer programs available for detection of singular lines[4].

The results obtained in Section 4 and 5 of this work led to the thought that singular lines on systems with two variable parameters can be considered as special cases of a system with three variable parameters in which one of the parameters is independent of the other two and has a fixed value. This value is obtained from one of the planes represented by equations 3.5. It is clear that the above mentioned plane is parallel to the one generated by the axes represented by the other two parameters.

Figures 6.2 and 6.3 will help to clarify the concept above stated. Figure 6.2 shows the general case where ζ and ω_n have been given and the point corresponding to the given value of S maps in the three-dimensional space as the line AB which is the locus of the values of α, β and γ that will produce the desired pair of roots with parameters for S : ζ and ω_n on the s-plane. Line AC is the projection of line AB onto the $\alpha\beta$ plane. Figure 6.3 shows the special case where plane A is parallel to the $\alpha\beta$ plane through $\gamma = \gamma_1$. The intercept of planes A and B is the line AB, that establishes the relation between α and β as a straight line on

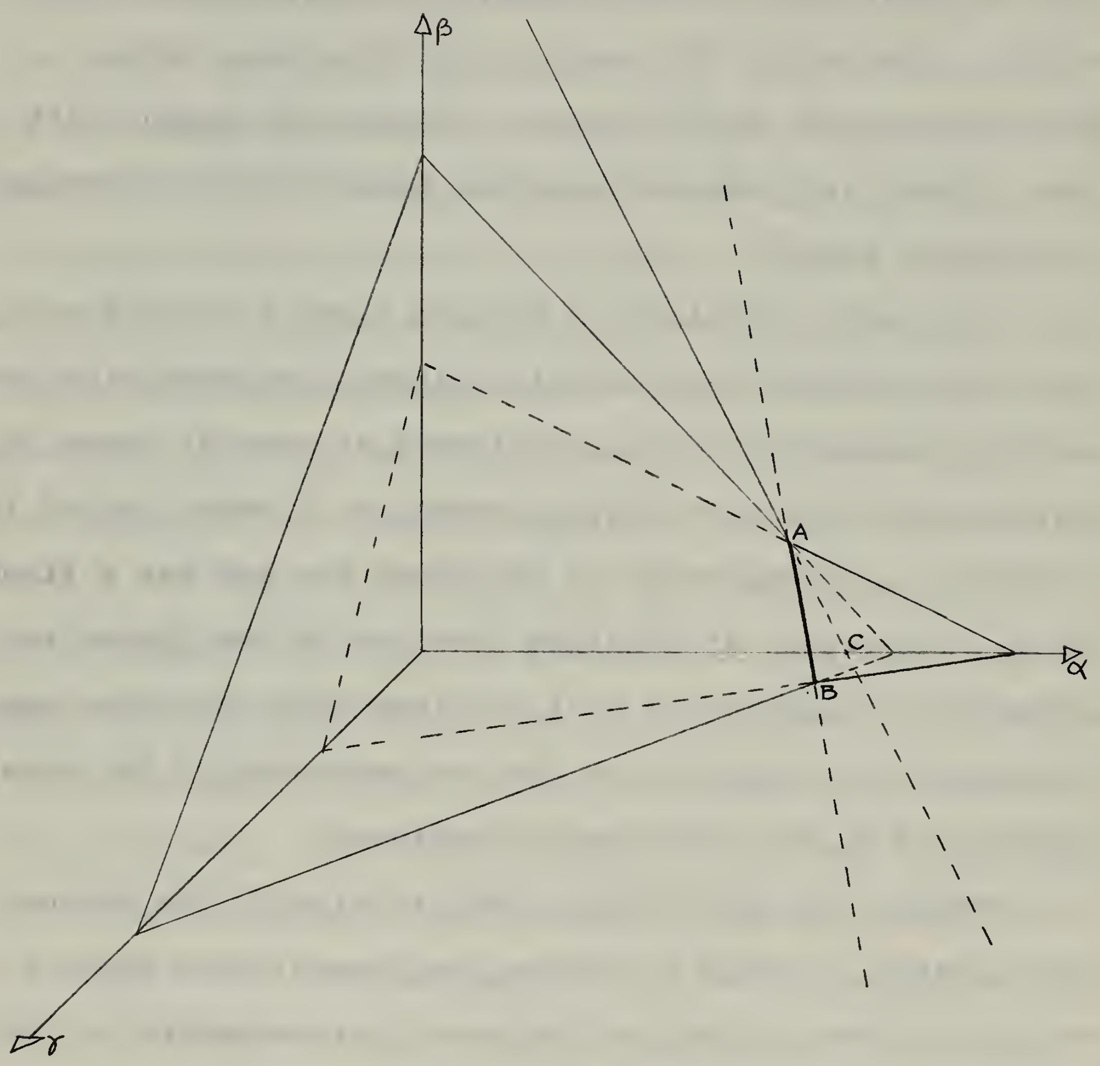


Fig. 6.2 - General case of mapping of a point on the s-plane into a straight line in the three dimensional space $\alpha\beta\gamma$.

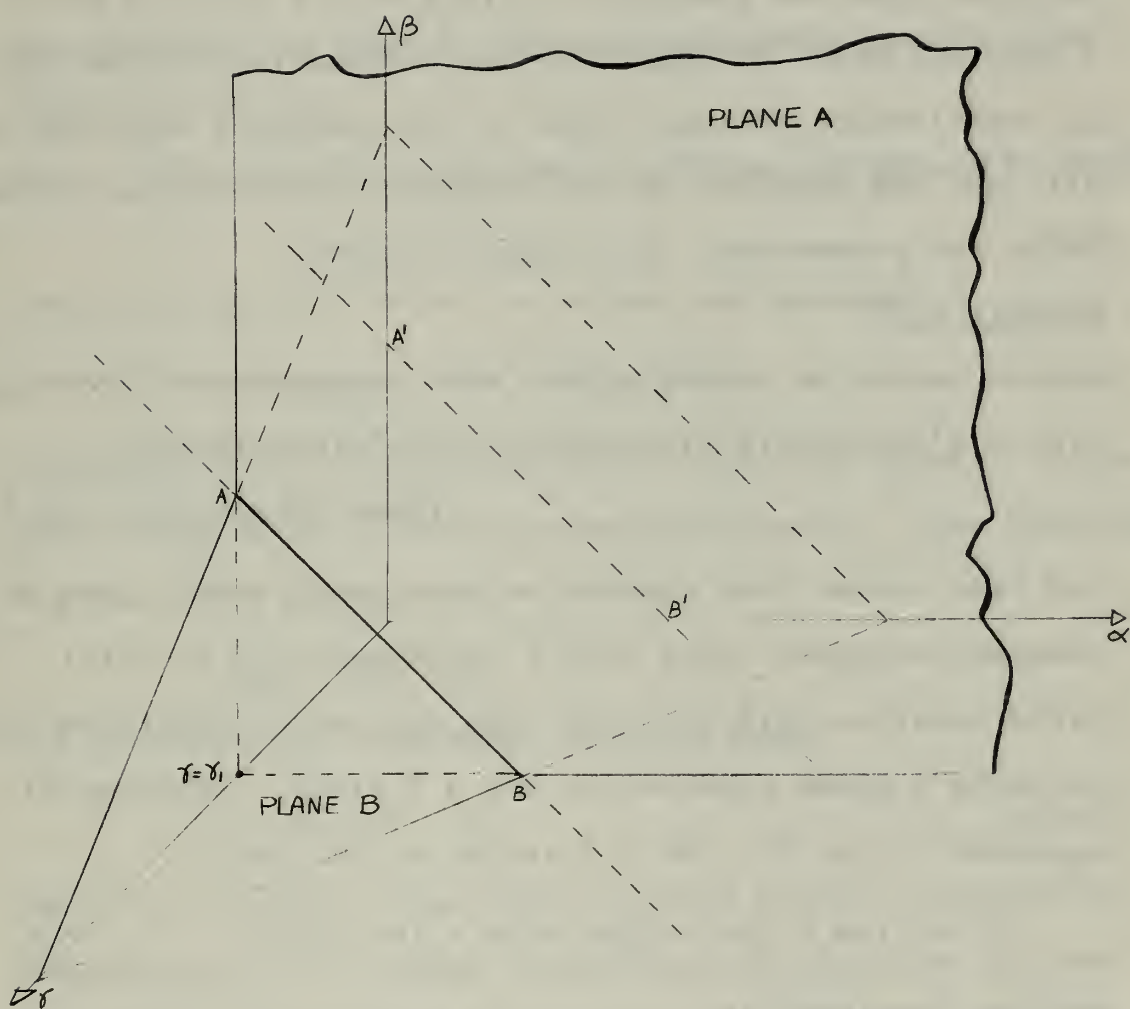


Fig. 6.3 - Special case of mapping of a point on the s-plane with generation of singular line.

the α β plane that passes through $\gamma = \gamma_1$. (Plane A in this case). So much for the geometric interpretation; the mathematical interpretation in terms of the division method outlined in Section 5 is that from one of the two equations obtained from the remainder, $F_1(\alpha,\beta,\gamma) = 0$ or $F_2(\alpha,\beta,\gamma) = 0$ a specific value of one of the parameters is obtained and by substitution of this value in the remaining equation, we will get the equation of the straight line relating to the other two parameters. An example follows.

Example 6.1:

Let us assume we have a system with characteristic equation

$$G(S) = S^4+(10\alpha+10)S^3+(40\alpha+5\beta+10\gamma+30)S^2+(80\alpha+10\beta+252) +(70\alpha+10\beta+25\gamma+340) \qquad 6.5$$

and also assume that somehow we have found that a pair of complex conjugate roots with $\zeta = 0.5$ and $\omega_n = 2.0$ will yield equation $F_1(\alpha,\beta,\gamma) = 0$ from the set of equations 5.4, defining a plane parallel to the α β plane. Division of equation 6.5 by $S^2 + 2S + 4$ yields as quotient:

$$S^2 + (10\alpha + 8)S + (20\alpha + 5\beta + 10\gamma + 10) \qquad 6.6$$

and the remainder is

$$(-20\gamma + 200)S + (-10\alpha - 10\beta - 15\gamma + 300) \qquad 6.7$$

By letting the remainder equal to zero we get

$$\gamma - 10 = 0 \qquad 6.8a$$

$$10\alpha + 10\beta + 15\gamma - 300 = 0 \qquad 6.8b$$

The graphical representation of equations 6.8 is shown in figure 6.4. Now, by continuing, from equation 6.8a:

$$\gamma = 10 \qquad 6.9a$$

and from equation 6.8b, substituting this value of γ:

$$\alpha + \beta - 15 = 0 \qquad 6.9b$$

Now substituting back in the characteristic equation the value of γ we get

$$G_1(S) = S^4 + (10\alpha+10)S^3 + (40\alpha+5\beta+130)S^2 + (80\alpha+10\beta+252)S + (70\alpha+10\beta+590) \qquad 6.10$$

and dividing by $S^2 + 2S + 4$ we get the quotient

$$S^2 + (10\alpha+8)S + (20\alpha + 5\beta + 110) \qquad 6.11$$

Expression 6.11 is the same as 6.6, had the value of γ been substituted back in this expression directly. The remainder for the above division is:

$$-10\alpha - 10\beta + 150$$

and by making the remainder equal to zero:

$$\alpha + \beta - 15 = 0 \qquad 6.12$$

This is a singular line case in the $\alpha\ \beta$ plane, because by assigning to γ the value of 10 there are only two degrees of freedom left: α and β in equation 6.10. According to what was stated in Section 5 and taking $G_1(S)$ of equation 6.10 as the characteristic equation of the system with two variable parameters and using the two degrees of freedom in setting the value of a pair of complex conjugate roots $S = -1.0 \pm 1.732j$, the remainder of the division should be a set of two linear simultaneous equations with a unique solution for α and β.

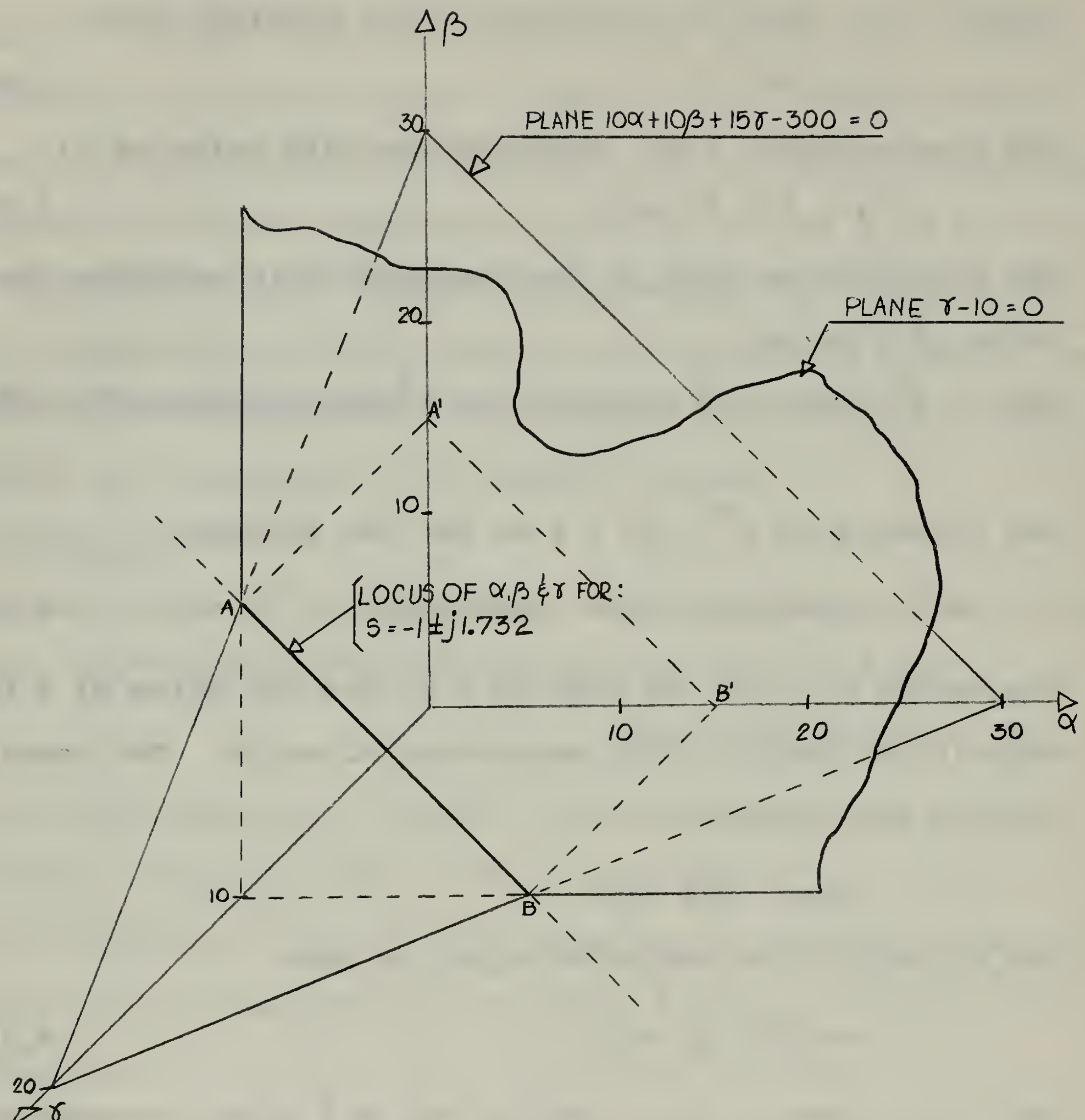


Fig. 6.4 - Singular line case for system represented by Equation 6.5 - Example 6.1.

Equation 6.12 is the only requirement for the remainder to be zero and therefore an infinite number of sets of values of α and β can be assigned. The root locus for the remaining roots of the polynomial is shown in figure 6.5, with α as the free parameter. This was obtained by substitution of

$$\beta = 15 - \alpha \qquad \text{(Equation 6.9b)}$$

into equation 6.11; then:

$$S^2 + (10\alpha + 8)S + (15\alpha + 185) = 0 \qquad 6.13$$

or

$$\frac{10\alpha(S + 1.5)}{(S^2 + 8S + 185)} = -1$$

If $10\alpha = K$ then

$$\frac{K(S+1.5)}{(S+4-j13)(S+4+j13)} = -1$$

By inspection of the root locus, and with the criterion of existence of dominance being that the real part of all of the remaining roots of the polynomial must be greater or equal to 10 times the real part of $S^2 + 2S + 4$, it can be seen that the root locus enters the desired region at point A and a section of it leaves the region at point B. Therefore, for values of α between these two regions we have acceptable dominance conditions. For point A the value of α is 2.41. By setting α:

$$\alpha = \frac{\alpha B + \alpha A}{2} = \frac{3.61}{2} = 1.80$$

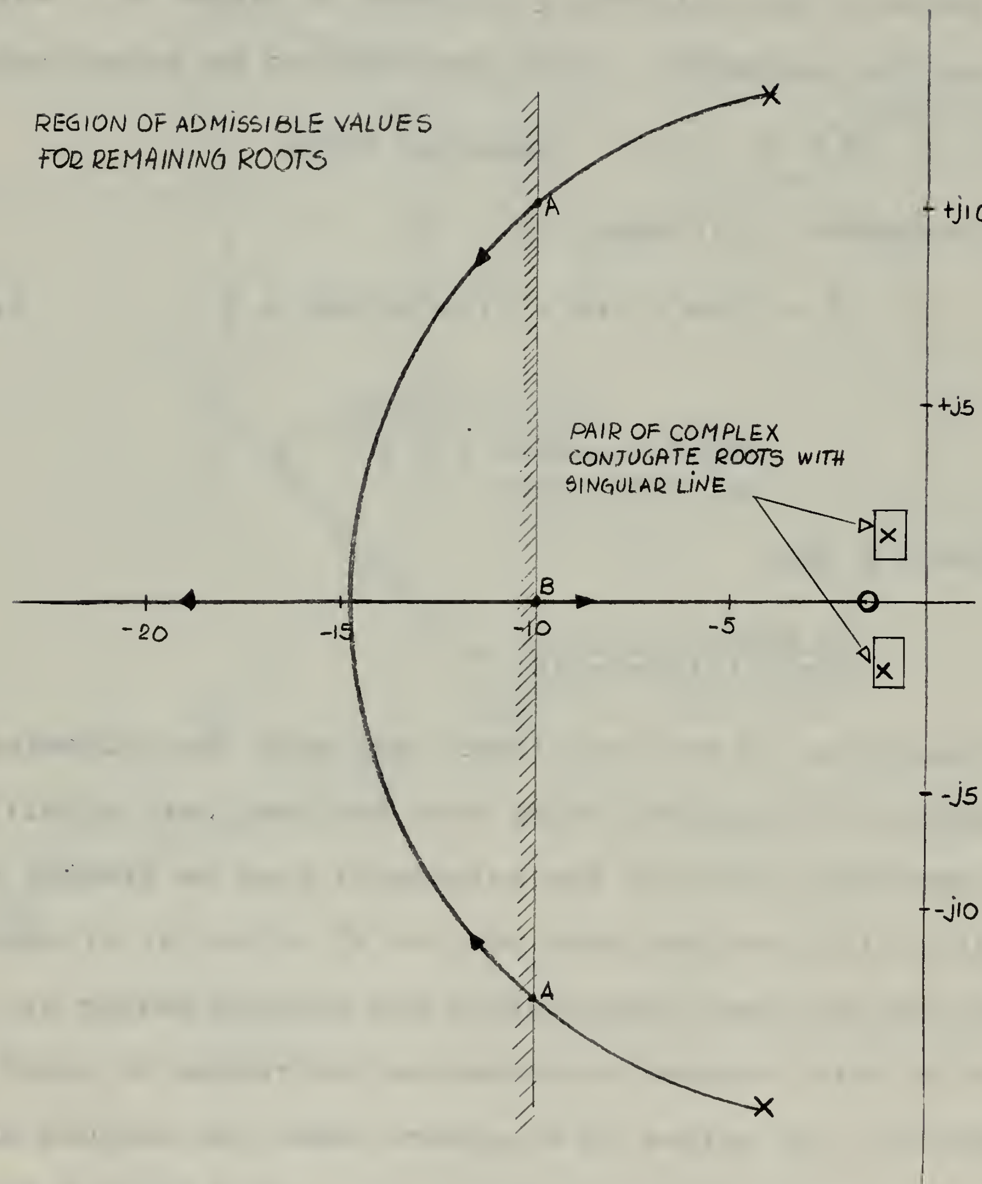


Fig. 6.5 - Modified root locus α as free parameter.
Example 6.1 - Singular line case.

we can have tolerances for α of $\pm$ 33% from the chosen value. Then for β we get:

$$\beta = 13.19$$

with tolerance for β of $\pm$ 4.65% from the assigned value.

It has been proven that the singular line in the α β plane can be taken as a special case of the three-parameter case. Unfortunately, the detection of planes parallel to either of the planes containing the coordinate axes is a job as difficult as the detection of singular line in the α β case. From the analytic point of view it will be possible to modify the coefficients of the different powers of S in such a way that a plane originally at an angle with the planes containing the coordinate axes becomes parallel to one of them. But from the practical point of view it is difficult to get this modification into the system, because these coefficients come from manipulation on the block diagram and usually affect several terms of the characteristic equation. Another limitation may be that the desired values are not physically attainable, due to the intrinsic composition of the elements of the system. From preceeding sections it has been known that a point on the s-plane maps in a straight line in the α β γ space and this straight line has a projection on the α β plane. Therefore, it might be easier from the designer's point of view, to "simulate" a singular line case by using three variable parameters, and instead of one set-up for adjustment of parameters, use two; that is: for each change in say, γ, it would be necessary to modify accordingly α and β. The expressions that will determine the modification procedure are precise mathematical expressions represented by equations 5.4 and they can be simulated by analog or digital means available. However, there are considerations such as weight, space and economy that will put a burden on a design of this type, but in the event that they can be overlooked, we will gain the ability of choosing the pair of dominant poles that will help in the achievement of the performance specifications and obtaining solid tolerance limits on the parameters. Engineering judgement must be used in this case to weight all these considerations.

When the design considerations stated above become a real problem and a singular line case is desirable, it is worth while to study the coefficients of the polynomial which represents the characteristic equation, and by algebraic manipulation, obtain the singular line case. The method outlined in the following section for a fourth order polynomial can be used in higher-order polynomials. The results obtained are quite satisfactory and a clear picture of what has to be done to obtain singular lines is presented.

6.1 GENERATION OF SINGULAR LINES IN A FOURTH-ORDER POLYNOMIAL WITH THREE VARIABLE PARAMETERS.

A Fourth-Order characteristic equation of a system with three variable parameters can be written in general form as:

$$S^4 + (b_3\alpha+c_3\beta+e_3\gamma+f_3)S^3 + (b_2\beta+c_2\beta+e_2\gamma+f_2)S^2 + (b_1\alpha+c_1\beta+e_1\gamma+f)S + (b_o\alpha+c_o\beta+e_o\gamma+f_o) = 0 \quad 6.14$$

By making the substitution of

$$d_k = e_k\gamma + f_k \quad 6.15$$

equation 6.14 can be written as:

$$S^4 + (b_3\alpha+c_3\beta+d_3)S^3 + (b_2\alpha+c_2\beta+d_2)S^2 + (b_1\alpha+c_1\beta+d_1)S + (b_o\alpha+c_o\beta+d_o) = 0 \quad 6.16$$

It is clear that the case of singular lines will depend on the configuration of the polynomial, that is, on the coefficients of the different powers of S. Therefore, it is possible to determine what coefficients must be affected in order to obtain singular lines. If it is assumed that the polynomial has singular lines for values of $x = 2\zeta\omega_n$ and $y = \omega_n^2$, division of the polynomial by: $S^2 + x\,S + y$ yields the follwoing results:

Quotient:

$$S^2 + (b_3\alpha+c_3\beta+d_3-x)S + (b_2-xb_3)\alpha + (c_2-xc_3)\beta + (d_2-y-xd_3+x^2) \quad 6.17$$

The remainder is:

$$F_1(\alpha,\beta)S + F_2(\alpha,\beta) \qquad 6.18$$

where:

$$F_1(\alpha,\beta) = (b_1-yb_3-xb_2+x^2b_3)\alpha + (c_1-yc_3-xc_2+x^2c_3)\beta + (d_1-yd_3+2xy-xd_2+x^2d_3-x^3) \qquad 6.19$$

and

$$F_2(\alpha,\beta) = (b_o-yb_2+xyb_3)\alpha + (c_o-yc_2+xyc_3)\beta + (d_o-yd_2+y^2+xyd_3-yx^2) \qquad 6.20$$

Singular lines can be generated for each of the following cases:

<u>Case I</u>: $F_1(\alpha,\beta) = 0$ for arbitrary values of α and β. Then:

$$b_1 - yb_3 - xb_2 + x^2b_3 = 0 \qquad 6.21$$

$$c_1 - yc_3 - xc_2 + x^2c_3 = 0 \qquad 6.22$$

$$d_1 - yd_3 + 2xy - xd_2 + x^2d_3 - x^3 = 0 \qquad 6.23$$

which is a system with three equations and two unknowns (x and y). From inspection of the system the following conclusions are drawn:

a). It is possible to solve for x and y by utilizing only equations 6.21 and 6.22.

b). Equation 6.23 is a function only of d_1, d_2 and d_3, which in turn are functions of γ (equation 6.15). Therefore, once the values of x and y are found, γ can be set at the value that will satisfy equation 6.23 for the values obtained for x and y. It is worth noting that the modifications in equation 6.23 do not affect the values of x and y obtained before. Then, from equations 6.21 and 6.22:

$$2\zeta\omega_n = x = \frac{c_1 b_3 - b_1 c_3}{c_2 b_3 - b_2 c_3} \qquad 6.24$$

$$\omega_n^2 = y = \frac{b_1 - x b_2 + x^2 b_3}{b_3}$$

and the third equation to be satisfied once x and y have been found is:

$$d_1 - y d_3 + 2xy - x d_2 + x^2 d_3 - x^3 = 0 \qquad 6.25$$

By substituting $d_k = e_k \gamma + f_k$ and using the values obtained for x and y we can solve for γ.

If the values of ζ and ω_n are not satisfactory to the designer, it is possible to know exactly what coefficients can be modified in order to get the desired values. In this case if for example $y = \omega_n^2$ is satisfactory but we wish to change $x = 2\zeta\omega_n$, then C_1, C_2, b_1 and C_3 can be modified at will without disturbing the value of y. From the above coefficients the designer might be able to choose the most convenient methods for the acutal modification, such as feedback paths, series compensation devices, etc. The above considerations are applicable to cases II and III which follow.

Case II. $F_2(\alpha,\beta) = 0$ for arbitrary values of α and β. Then:

$$b_o - yb_2 + xyb_3 = 0$$
$$c_o - yc_2 + xyc_3 = 0 \qquad 6.26$$
$$d_o - yd_2 + y^2 + xyd_3 - yx^2 = 0$$

From the first two equations we get:

$$2\zeta\omega_n = x = \frac{b_oc_2 - b_2c_o}{b_oc_3 - b_3c_o}$$
$$\qquad 6.27$$
$$\omega_n^2 = y = \frac{b_o}{b_2 - xb_3}$$

and

$$d_o - yd_2 + y^2 + xyd_3 - yx^2 = 0 \qquad 6.28$$

Again, by substitution of $d_k = e_k\gamma + f_k$, and using the values obtained for x and y, it is possible to solve for γ.

Case III. $F_1(\alpha,\beta) = F_2(\alpha,\beta)$ for arbitrary values of α and β. Then:

$$b_1 - yb_3 - xb_2 + x^2b_3 = b_o - yb_2 + xyb_3$$
$$c_1 - yc_3 - xc_2 + x^2c_3 = c_o - yc_2 + xyc_3 \qquad 6.29$$
$$d_1 - yd_3 + 2xy - xd_2 + x^2d_3 - x^3 = d_o - yd_2 + y^2 + xyd_3 - yx^2$$

From the first two equations we get

$$2\zeta\omega_n = x = \frac{(b_2 - b_3)(c_1 - c_o) + (b_1 - b_o)(c_3 - c_2)}{b_3(c_1 - c_2 - c_o) + c_3(b_2 - b_1 + b_o)}$$
$$\qquad 6.30$$
$$\omega_n^2 = y = \frac{b_1 - xb_2 + x^2b_3 - b_o}{b_3 + xb_3 - b_2}$$

and

$$d_1 - yd_3 + 2xy - xd_2 + x^2 d_3 - x^3 = d_o - yd_2 + y^2 + xyd_3 - yx^2 \qquad 6.31$$

That with the given values of x and y will yield the value of γ.

The important feature of the results stated above is that the designer knows exactly what coefficients of the characteristic equation affect directly the values of ζ and ω_n and hence, what parts of the actual system must be modified in order to obtain the desired values.

The solutions proposed above are for the case where the "free parameter" is γ; that is: one of the planes considered in Section 6 is parallel to the $\alpha\beta$ plane through the calculated value of γ. The quotient (equation 6.17) will be then a function of α and β. From the remainder, α can be expressed as a function of β, or β as function of α, and root locus techniques applied in order to determine the location of the remaining roots of the polynomial. A sensitivity study can be made on the root locus. For applications in self-adaptive systems there will be need of only one set-up for modifications when changes in either of the parameters occur. The following example will illustrate the technique.

<u>Example 6.2</u>:

A system has the following characteristic equation:

$$S^4 + (10\alpha+10)S^3 + (40\alpha+5\beta+130)S^2 + (80\alpha+10\beta+10\gamma+5)S + (70\alpha+10\beta+\gamma+200) \qquad 6.32$$

Question: What values of ζ and ω_n are attainable in singular line cases and what are the required values of γ?

For Case I, substituting the values of the coefficients in equations 6.24:

$$x = 2\zeta\omega_n = 2 \quad \text{or} \quad \zeta = 0.5$$

$$y = \omega_n^2 = 4 \quad \text{or} \quad \omega_n = 2 \tag{6.33}$$

$$\gamma = 24.7$$

Substituting the value of γ in equation 6.32 and dividing by $S^2 + 2S + 4$ we get:

Quotient:

$$S^2 + (10\alpha+8)S + (20\alpha+5\beta+110) = 0 \tag{6.34}$$

Remainder:

$$-10\alpha - 10\beta - 215.3 = 0 \tag{6.35}$$

Now root locus techniques can be applied to equation 6.34 after substituting β as a function of α from equation 6.35. The expressions for the quotient and the remainder can also be obtained from equations 6.17 and 6.20 simply by substituting the values of the coefficients.

For Case II, substituting the values of the coefficients in equations 6.27:

$$x = 2\zeta\omega_n = 0.5 \quad \text{or} \quad \zeta = 0.1765$$

$$y = \omega_n^2 = 0.2 \quad \text{or} \quad \omega_n = 1.4142 \tag{6.36}$$

$$\gamma = 46.5$$

Substituting the value of the γ in equation 6.32 and dividing by $S^2 + 0.5S + 2$ we get:

Quotient:

$$S^2 + (10\alpha+9.5)S + (35\alpha+5\beta+123.25) = 0 \qquad 6.37$$

Remainder:

$$(42.5\alpha + 7.5\beta + 389.375)S = 0$$

For Case III, substituting the values of the coefficients in equations 6.30:

$$2\zeta\omega_n = x = 1 \quad \text{or} \quad \zeta = 0.5$$

$$\omega_n^2 = y = 1 \quad \text{or} \quad \omega_n = 1 \qquad 6.38$$

$$\gamma = 204/9 = 22.667$$

Division yields

Quotient:

$$S^2 + (10\alpha+9)S + (30\alpha+5\beta+120) = 0 \qquad 6.39$$

Remainder:

$$(40\alpha+5\beta+102.667)S + (40\alpha+5\beta+102.667) = 0 \qquad 6.40$$

7. CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER WORK

In this work it is shown that the increase in degrees of freedom in a system really helps the designer in the task of obtaining from it the desired characteristics. Mitrovic's approach has proved to be useful in the parameter plane or when products of parameters are present in the characteristic equation of a system. However, when the number of variable parameters is three or more it is difficult to obtain graphical information. The tabulated data obtainable from a computer comes only in quantized form and it is difficult for the designer to use it in seeking his objective, which is to make the best choice of the variable parameters. In the field of singular lines a method of generation of singular lines for a fourth-order polynomial was established and it is expected that formulae can be obtained for systems with characteristic equation of higher order. With this method the values of ζ and ω_n at which singular lines are expected and the value at which a third parameter must be set are obtained. In addition, the modifications in the system for which the values of ζ and ω_n are changed are indicated.

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13. ABSTRACT

A study of Dynamic Systems with three variable parameters is made by an initial scanning of the basic geometric properties on the three dimensional space generated by these parameters. From these geometric properties, a root locus technique that simplifies greatly the amount of work in analysis and design is developed. This technique is extended to systems with k variable parameters. Finally, singular lines in the parameter plane are treated as a special case of the parameter space, and formulae are derived for a fourth-order system leaving the field open for systems of higher order.

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